Von Wright’s truth-logic and around
ALEXANDER S. KARPENKO

Abstract. In this paper von Wright’s truth-logic $T''$ is considered. It seems that it is a De Morgan four-valued logic $DM4$ (or Belnap’s four-valued logic) with endomorphism $e_2$. In connection with this many other issues are discussed: twin truth operators, a truth-logic with endomorphism $g$ (or logic $Tr$), the lattice of extensions of $DM4$, modal logic $V2$, Craig interpolation property, von Wright–Segerberg’s tense logic $W$, and so on.

Keywords: Wright’s truth-logic, De Morgan four-valued logic, twin truth operators, tetravalent modal logic TML, truth logic $Tr$, modal logic $V2$, von Wright–Segerberg’s tense logic

1 Four-valued classical logic $C_4$ and four-valued De Morgan logic $DM4$

Let $M_4^C$ be a four-valued logical matrix

\[
M_4^C = \langle \{1, b, n, 0\}, \supset, \lor, \land, \neg, \{1\} \rangle
\]

which is obtained from the direct product of the matrix $M$ (for classical propositional logic $C_2$) with itself, i.e. $M_4^C = M_2^C \times M_2^C$, where matrix operations $\supset, \lor, \land, \neg$ are the following:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\neg x$</th>
<th>$\supset$</th>
<th>$1$</th>
<th>$b$</th>
<th>$n$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b$</td>
<td>$n$</td>
<td>$b$</td>
<td>1</td>
<td>1</td>
<td>$n$</td>
<td>0</td>
</tr>
<tr>
<td>$n$</td>
<td>$b$</td>
<td>$n$</td>
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<td>0</td>
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</tbody>
</table>
Note that the set of truth-values \{1, b, n, 0\} is partially-ordered in the form \(0 < n, b < 1\), i.e. \(n\) and \(b\) are incomparable. As usual,

\[
x \lor y =: \neg x \supset y,
\]

\[
x \land y =: \neg(\neg x \lor \neg y),
\]

\[
x \equiv y =: (x \supset y) \land (y \supset x).
\]

It is well known that matrix \(M_4^C\) is characteristic for calculus \(C_2\). The logic with the above operations is denoted as \(C_4\). As usual, we will denote connectives and the similar operations by the same symbols.

Then the logic with the operations \(\lor, \land\) and \(\sim\) is called four-valued De Morgan logic \(DM_4\), where \(\sim\) is De Morgan negation: \(\sim 1 = 0, \sim b = b, \sim n = n, \sim 0 = 1\) (see \[5\], \[9\]). In another terminology, \(DM_4\) is Belnap’s four-valued logic \[3\].

2 Endomorphismus in the distributive lattices

In \[6\] the authors point out the fact that the modal and tense operations in a number of modal and tense logics and in corresponding algebras are expressed in terms of endomorphismus in the distributive lattices.

Let us consider one-place operations \(g, e_1\) and \(e_2\)

\[
\begin{array}{c|cccc|c|cccc}
\lor & 1 & b & n & 0 \\
1 & 1 & 1 & 1 & 1 \\
b & 1 & b & 1 & b \\
n & 1 & 1 & n & n \\
0 & 1 & b & n & 0 \\
\hline
\land & 1 & b & n & 0 \\
1 & 1 & b & n & 0 \\
b & b & b & 0 & 0 \\
n & n & 0 & n & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(g(x))</th>
<th>(e_1(x))</th>
<th>(e_2(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>n</td>
<td>0</td>
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<tr>
<td>n</td>
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which are the endomorphismus in the distributive lattices:

\[ f(x \lor y) = f(x) \lor f(y), \quad f(x \land y) = f(x) \land f(y), \]
\[ f(\neg x) = \neg f(x), \quad f(1) = 1, \quad f(0) = 0, \quad f(x^\delta) = (f(x))^\delta, \]

where \( f \) can be any operations from \( g, e_1 \) and \( e_2 \).

3 Von Wright’s truth-logic \( T'' \)

Now in the new terms introduced above we can define Wright’s truth-logic. The expansion of \( \text{DM4} \) by the endomorphism \( e_2 \) leads to the logic which G.H. von Wright in 1985 denoted as \( \text{T''LM} \) and called a ‘truth-logic’ (see [28]). For the sake of brevity, we will denote it as \( T'' \). Here a truth-operator \( T \) is the endomorphism \( e_2 \). Note that the following important definitions hold:

\begin{align*}
(\ast) \quad e_1(x) &=: \sim (e_2(\sim x)) \quad \text{and} \quad e_2(x) =: \sim (e_1(\sim x)).
\end{align*}

It is easy to show that all four-valued \( J_i(x) \)-operations are definable in \( \text{T''LM} \), where

\[ J_i(x) = \begin{cases} 
1, & \text{if } x = i \\
0, & \text{if } x \neq i \quad (i = 1, n, b, 0).
\end{cases} \]

Thus, we have:

\[
\begin{array}{c|c|c|c|c}
 x & J_1(x) & J_b(x) & J_n(x) & J_0(x) \\
\hline
 1 & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 0 & 0 \\
n & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[ x^\delta = \begin{cases} 
x, & \text{if } \delta = 1 \\
\neg x, & \text{if } \delta = 0.
\end{cases} \]

2In [19] a four-valued ‘logic of falsehood’ \( \text{FL4} \) is formalized. In our terms it is the expansion of the language of \( \text{DM4} \) by the endomorphism \( e_1 \). So, in virtue of (\ast) logics \( \text{FL4} \) and \( T'' \) are functionally equivalent.
One may easily verify that

\[ J_1 =: e_1(x) \land e_2(x), \]

\[ J_b =: \sim e_1(x) \land e_2(x), \]

\[ J_n =: e_1(x) \land \sim e_2(x), \]

\[ J_0 =: \sim e_1(x) \land \sim e_2(x). \]

Note that \( e_2(x) =: J_1 \lor J_b. \) Then Wright’s logic \( T'' \) is De Morgan logic \( \text{DM4} \) with all \( J_i(x) \)-operators (but, it is important, without classical negation \( \sim \)). Note also that in many finite modal logics the operator \( J_1 \) is the modal operator of necessity \( \square \). Then the well-known tetravalent modal logic \( \text{TML} \) is \( \text{DM4} \) with the operator \( \square \) added to its language (see especially [9]). So \( T'' \) is an extension of \( \text{TML} \).

Now we need some additional definitions. A finite-valued logic \( L_n \) with all \( J_i(x) \)-operators is called \( \text{truth-complete} \) logic, and a logic \( L_n \) is said to be \( \text{C-extending} \) iff in \( L_n \) one can functionally express the binary operations \( \supset, \lor, \land \), and the unary negation operation, whose restrictions to the subset \( \{0, 1\} \) coincide with the classical logical operations of implication, disjunction, conjunction, and negation. In virtue of result of [2] every truth-complete and \( \text{C-extending} \) logic has Hilbert-style axiomatization extending the \( \text{C}_2 \). It means that Wright’s \( T'' \) logic has such an axiomatization. Moreover, it follows from [1] that we have adequate first-order axiomatization for logic \( T'' \) with quantifiers.

It is very interesting to generalize given four-valued von Wright’s logic, i.e. to consider an arbitrary finite-valued De Morgan logic with all \( J_i(x) \)-operators. As a result, we obtain an entirely new class of many-valued logics which I suggest to call ‘Wright’s many-valued logics’ and a new class algebras which I suggest to call ‘Wright’s algebras’. Then again it follows from [1] that for such logics we have adequate first-order axiomatization.

\[ ^3\text{However, see also [5].} \]
4 Properties of a truth-operator $T$ and the twin truth operators

The following two properties of a truth-operator $T$ are useful:

(I) $T(\sim x) \equiv \sim T(x)$

(II) $T(x) \lor T(\sim x)$ — the law of excluded middle.

Note that these two conditions are required in the Tarski’s axiomatic theory of truth with a predicate symbol $True$ (see [12]).

None of these conditions is fulfilled in the logic $T''$. However it is interesting to consider the operations $e_1$ and $e_2$ as the twin truth operators $T_1$ and $T_2$ bearing in mind ($\ast$). Then

(I') $T_1(\sim x) \equiv \sim T_2(x)$

(II') $T_1(x) \lor T_2(\sim x)$ — the law of excluded middle.

Here we must note that the main goal pursued by von Wright has been the construction of paraconsistent logic. So the choice of the operations $\sim$ and $T_2$ is such that the law of contradiction

$\sim (T_2(x) \land T_2(\sim x))$

is not valid in $T''$. But it is interesting that this law is valid in the form

$\sim (T_1(x) \land T_2(\sim x))$ or $\sim (T_2(x) \land T_1(\sim x))$.

We want to stress that von Wright’s truth logic with the twin truth operators $T_1$ and $T_2$ seems to us very interesting.

5 Logic $\text{Tr}$

Let us consider the expansion of $\text{DM4}$ by the endomorphism $g$. Now the conditions (I)–(II) are fulfilled. Note that operators $\sim$ and $g$ commute among themselves, i.e.

$\sim g(x) \equiv g \sim (x)$. 
Moreover, this allows to define the classical negation $\neg$:

$$\neg(x) =: \sim g(x).$$

We denote a truth logic with the set of operations $\{\lor, \land, \sim, g\}$ by $\text{Tr}$.

There is a very simple and nice axiomatization of this logic (see justification below), where the operation $T$ is $g$:

(A0) Axioms of classical propositional logic $\mathbf{C}_2$.

(A1) $T(A \supset B) \equiv (TA \supset TB)$.

(A2) $\neg TA \equiv T \neg A$.

(A3) $TTA \equiv A$.

The single rule of inference: *modus ponens*.\(^4\)

It is worth to mention that there is a generalized truth-value space in kind of *bilattice* (see [11]). Indeed, smallest nontrivial bilattice is just the four-valued Belnap’s logic. In [8] M. Fitting extends a first-order language by notation for elementary arithmetic, and builds the theory of truth based on bilattice. This four-valued theory of truth is an alternative to Tarsky’s approach.

Also in one case, Fitting extends this language by the operation ‘*conflation*’ (endomorphism $g$).

### 6 Interrelations between $T''$ and $\text{Tr}$

Let $P_4$ be Post’s four-valued functionally complete logic (see [20]). The set operation $R$ is called functionally *precomplete* in $P_4$ if every enlargement $\{R, f\} = R \cup \{f\}$ of the set $R$ by an operation $f$ such that $f \notin R$ and $f \in P_4$ is functionally complete.

It is not difficult to prove, that the logic with the set of the operations $\{\lor, \land, \sim, e_2, g\}$ is four-valued Lukasiewicz logic $\mathbf{L}_4$ which first appeared in [15]). According to Finn’s result $\mathbf{L}_4$ is precomplete in $P_4$ (see [4]). Note that in $\mathbf{L}_4$

\(^4\)At the time of my report G. Sandu had asked about the logic $\text{Tr}$ with the axiom $(A4) TA \equiv A$. Let’s denote this logic by $\text{Tr}^c$. If we take the operation $T$ as identity operation of $\mathbf{C}_2$ then the logic $\text{Tr}^c$ is a conservative extension of $\mathbf{C}_2$.\(^4\)
Von Wright’s truth-logic and around 45

\( x \lor y = \max(x, y) \) and \( x \land y = \min(x, y) \),
i.e. the truth-values in \( L_4 \) are linearly-ordered\(^5\).

As a result, we have the following lattice of extensions of \( DM_4 \):

\[ \text{DM}_4 \rightarrow \text{T'} \rightarrow L_4 \rightarrow \text{Tr} \]

7 Modal logic V2

In [25] Sobociński presents the formula (\( \beta_2 \)):

\[ \Box p \lor \Box(p \supset q) \lor \Box(p \supset \neg q). \]

He establishes that it is not provable in \( S_5 \), and \( S_5 \) plus (\( \beta_2 \)) is not classical calculus \( C_2 \). In [26] this logic is denoted by \( V_2 \). As a consequence of Scroggs’ result about *pretabularity* of \( S_5 \)\(^6\) logic \( V_2 \) is finite-valued one. It was remarked that four-valued matrix of

\[ \{1, b, n, 0\}, \supset, \neg, \Box, \{1\} \],

is characteristic for \( V_2 \) (see e.g. [5, p. 190]).

In [6] it has been shown that logics \( Tr \) and \( V_2 \) are functionally equivalent:

\(^5\)In details about different finite-valued logics see in [13, ch. 5].

\(^6\)A logic \( L \) is said to be *pretabular* if it is not finite (tabular), but its proper extension is finite. Scroggs [22] has shown that \( S_5 \) has no finite characteristic matrix but every proper normal extension does.
Note that in [5] an algebraic semantics (named to MB-algebras) has been developed for logic Tr (V2). MB-algebra is an expansion of De Morgan algebra by Boolean negation ¬. In this case \( g(x) = \neg \sim (x) = \neg \sim x \). It is interesting that Pynko [21] introduces a similar algebraic structure called De Morgan boolean algebra. He also suggests Gentzen-style axiomatization of four-valued logic denoted by DMB4.

In [17] Maksimova considers all normal extensions of modal logic S4 with the Craig interpolation property. From this it follows that modal logic V2 is the single normal extension of modal logic S5 with the Craig interpolation property (between S5 and C2). Since the logics Tr and V2 are functionally equivalent then the following theorem can be proved:

**Theorem 1.** A logic Tr has the Craig interpolation property.

### 8 Von Wright–Segerberg’s tense logic W

It is interesting that we can come to the logic Tr on the basis of an entirely different considerations. In [27] von Wright presents a tense logic ‘And next’ which deals with discrete time. In [23] Segerberg reformulates it under the name W and provides other proofs of completeness theorem, and decision procedure.\(^8\)

A logic W is a very simple propositional logic in which a new unary operation S with the intuitive meaning of ‘tomorrow’ is added to the language of the classical propositional calculus. W is axiomatized in the following way:

- **(A0)** Axioms of classical propositional logic C2.
- **(A1)** \( S(A \supset B) \equiv (SA \supset SB) \).
- **(A2)** \( \neg SA \equiv S\neg A \).

\(^7\)However, see [23, p. 49].

\(^8\)For detailed overview of von Wright’s tense logic see Segerberg’s paper [24].
The rules of inference:

R1. *Modus ponens*,

R2. *From A follows SA*.

Segerberg suggests the following Kripke-style semantics for \( W \) (this semantics in a simplified way is presented in [7, p. 288]). Let \( N = 0, 1, 2, \ldots \) be the set of possible worlds. Valuation \( v(p_i, w) = 1, 0 \) (‘truth’, ‘falsehood’) for propositional variables \( p_i \) and \( w \in N \). For \( \supset \) and \( \neg \) as usual, and for \( SA : v(SA, w) = v(A, w + 1) \). Pay attention that \( W \) is the logic that defines the set formulas valid in \( N \).

Concerning the logic \( W \) there are the following meta-logical results:

1) There is no finite axiomatization of \( W \) with modus ponens as sole inference rule [23].

2) Logic \( W \) is pretabular [7].

It is worth emphasizing that in [6] Mučnik has devised algebraic semantics for \( W \), named \( Bg \)-algebras, and has proved Stone’s representation theorem for them. Here it is noted that \( Bg \)-algebra with involution, where \( gg(x) = x \), corresponds to the logic \( V2 \). Thus we again have come to the logic \( Tr \).

Note than in [18] Kripke frame, consisting two possible worlds, is represented for \( V2 \). Here we describe Kripke frame \( i =< T, R > \) for \( W \) and \( Tr \), where \( T \) is the set of instants of time.

A Kripke frame \( i =< T, R > \) is a frame for \( W \) if the following conditions fulfill:

1. \( \forall w \in T \exists v \in TwRv \)

   ‘from every point (instant) something is accessible’.

2. \( \forall w \in T \forall v_1 \in T \forall v_2 \in T (wRv_1 \& wRv_2 \Rightarrow v_1 = v_2) \)

   ‘from every point no more than one point is accessible’.

And for \( Tr \) it is necessary to add:
3. ∀w₁ ∈ T ∀w₂ ∈ T ∀w₃ ∈ T(w₁Rw₂ & w₂Rw₃ ⇒ w₃ = w₁)

‘from every point in two steps we once again find ourselves in the same point’.

**Theorem 2.** Logic W + axiom (A3) SSA ≡ A and logic Tr are the same as the sets of derivable formulas.

**References**


