Equality of consequence relations in finite-valued logical matrices

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Abstract. In this paper the procedure is presented that allows to determine in finite number of steps if consequence relations in two finite-valued logical matrices for propositional language $L$ are equal.

Keywords: product of logical matrices, consequence relation, equality of matrices

In his paper ‘A test for the equality of truth-tables’ [2], J. Kalicki has described a general method for testing the equality of the classes of tautologies in different finite-valued matrices. Below I present a generalization of Kalicki’s method which allows to test whether the consequence relations in two finite-valued logical matrices are equal.

First, the question of equality of consequence relations in two arbitrary matrices will be reduced to the question of the properties of a single matrix. This matrix will be obtained from initial matrices via the operation of product, but it will have four classes of truth-values instead of the standard two (designated and non-designated). On the basis of these four classes I will define several consequence relations. The properties that these relations display in the product matrix will define if two initial matrices are equal in terms of consequence relation. Then I will show that it is sufficient to consider a finite set of formulas to investigate the properties in question, and that therefore a finite number of steps is required to determine if consequence relations are equal in two finite-valued matrices.

Let us begin with some necessary definitions.
Definition 1. A logical matrix is a structure $\mathfrak{M} = < V, F, D >$, where $V$ is the set of truth-values, $F$ is a set of functions on $V$ called basic functions, and $D$ is a designated subset of $V$.

In this paper we will only consider the logical matrices where $V$ is finite.

If for any $n$ it is true that $\mathfrak{M}$ contains as much $n$-ary elements of $F$ as there are $n$-ary connectives in some propositional language $L$, $\mathfrak{M}$ is a logical matrix for $L$. In that case we can establish a one-to-one correspondence between the elements of $F$ and the connectives of $L$, and define a valuation of a formula in $\mathfrak{M}$.

Definition 2. A valuation $v$ of formula $A$ in $\mathfrak{M}$ is a homomorphism of $L$ in $< V, F >$ such that

1. if $A$ is a propositional variable, then $v(A) \in V$;

2. if $A_1, A_2, \cdots, A_n$ are formulas, and $C$ is an $n$-ary connective of $L$, then $v(C(A_1, A_2, \cdots, A_n)) = f^n(v(A_1), v(A_2), \cdots, v(A_n))$, where $f^n$ is a function from $F$ corresponding to $C$.

The definition of consequence relation in $\mathfrak{M}$ is a standard one.

Definition 3. $\Gamma \models (\mathfrak{M})B$ iff there is no valuation $v$ in $\mathfrak{M}$, such that $v[\Gamma] \subseteq D(\mathfrak{M})$ (i.e. every formula from $\Gamma$ assumes a truth-value designated in $\mathfrak{M}$), and $v(A) \notin D(\mathfrak{M})$.

Let us denote as $C(\mathfrak{M})$ a set of ordered pairs $< \Gamma, B >$, such that $\Gamma$ is a set of formulas, $B$ is a formula, and $\Gamma \models (\mathfrak{M})B$. Now we will define the equality of consequence relations in two arbitrary matrices for $L$.

Definition 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the matrices for $L$. The consequence relations in $\mathfrak{A}$ and $\mathfrak{B}$ are equal iff $C(\mathfrak{A}) = C(\mathfrak{B})$.

Now we will make the transition from two matrices to one by applying the product operation. If $\mathfrak{A}$ and $\mathfrak{B}$ are the matrices for $L$, a one-to-one correspondence between the elements of their sets of basic functions can be established. This allows us to give the following definition.

Definition 5. A product of matrices $\mathfrak{A}$ and $\mathfrak{B}$ is a matrix $\mathfrak{C} = \mathfrak{A} \otimes \mathfrak{B}$, such that
• \( V(\mathcal{C}) \) is a Cartesian product of \( V(\mathcal{A}) \) and \( V(\mathcal{B}) \);

• for each pair of mutually corresponding \( k \)-ary basic functions \( f^k(x_1, x_2, \ldots, x_k) \) from \( \mathcal{A} \) and \( g^k(y_1, y_2, \ldots, y_k) \) from \( \mathcal{B} \) there is one and only one basic operation \( h^k \) from \( \mathcal{C} \), and \( h^k(< x_1, y_1 >, < x_2, y_2 >, \ldots, < x_k, y_k >) = < f^k(x_1, x_2, \ldots, x_k), g^k(y_1, y_2, \ldots, y_k) > \).

This is a standard product operation. However, the truth-values in \( \mathcal{C} \) will be divided into four classes\(^3\):

- \( < x_i, y_j > \in \omega(\mathcal{C}) \) iff \( x_i \in D(\mathcal{A}) \) and \( y_j \in D(\mathcal{B}) \);
- \( < x_i, y_j > \in \xi(\mathcal{C}) \) iff \( x_i \in D(\mathcal{A}) \) and \( y_j \notin D(\mathcal{B}) \);
- \( < x_i, y_j > \in \xi'(\mathcal{C}) \) iff \( x_i \notin D(\mathcal{A}) \) and \( y_j \in D(\mathcal{B}) \);
- \( < x_i, y_j > \in \phi(\mathcal{C}) \) iff \( x_i \notin D(\mathcal{A}) \) and \( y_j \notin D(\mathcal{B}) \).

I will now consider two definitions of consequence relation based on these four classes, \( \vdash_{\cup} \) and \( \vdash_{\cap} \).

**Definition 6.** \( \Gamma \vdash_{\cup} (\mathcal{C})B \) iff there is no valuation \( w \) in \( \mathcal{C} \), such that \( w[\Gamma] \subseteq \omega(\mathcal{C}) \), and \( w(A) \in \phi(\mathcal{C}) \).

**Lemma 1.** \( \Gamma \vdash_{\cup} (\mathcal{C})B \) iff \( \Gamma \vDash (\mathcal{A})B \) or \( \Gamma \vDash (\mathcal{B})B \).

**Proof.** (i) Let \( \Gamma \vdash_{\cup} (\mathcal{C})B \), and \( \Gamma \not\vDash (\mathcal{A})B \), and \( \Gamma \not\vDash (\mathcal{B})B \). Then there exists a valuation \( v^* \) in \( \mathcal{A} \), such that \( v^*[\Gamma] \subseteq D(\mathcal{A}) \) and \( v^*(A) \notin D(\mathcal{A}) \), and there exists a valuation \( u^* \) in \( \mathcal{B} \), such that \( u^*[\Gamma] \subseteq D(\mathcal{B}) \) and \( u^*(A) \notin D(\mathcal{B}) \). For every \( v \) and \( u \) there is a mapping \( w \) of the propositional variables of \( L \) on \( V(\mathcal{A}) \times V(\mathcal{B}) \), such that \( w(p_k) = v(p_k), w(p_k) > \), where \( p_k \) is a propositional variable. Obviously, every such \( w \) is a valuation in \( \mathcal{C} \). By definition of \( \mathcal{C} \), \( w^* \) obtained from \( v^* \) and \( u^* \) is such a valuation that \( w^*[\Gamma] \subseteq \omega(\mathcal{C}) \), and \( w^*(A) \in \phi(\mathcal{C}) \). That contradicts our assumption.

(ii) Let \( \Gamma \not\vdash_{\cup} (\mathcal{C})B \), and \( \Gamma \vDash (\mathcal{A})B \) or \( \Gamma \vDash (\mathcal{B})B \). Then there is a valuation \( w^* \) in \( \mathcal{C} \), such that \( w^*[\Gamma] \subseteq \omega(\mathcal{C}) \), and \( w^*(A) \in \phi(\mathcal{C}) \). For

\(^3\)This is essentially a distribution introduced by Kalicki [2], but he only needed three classes, so elements of \( \xi(\mathcal{C}) \) and \( \xi'(\mathcal{C}) \) were assigned to the same class.
every valuation \( w \) in \( \mathcal{C} \) there is the following valuation \( v \) in \( \mathfrak{A} \): if
\( w(p_k) = \langle x_i, y_j \rangle \), then \( v(p_k) = x_i \). By definition of \( \mathcal{C}, v^* \) obtained
this way from \( w^* \) is such a valuation in \( \mathfrak{A} \) that \( v^*[\Gamma] \subseteq D(\mathfrak{A}) \) and
\( v^*(A) \notin D(\mathfrak{A}) \). The reasoning for valuation \( u^* \) in \( \mathfrak{B} \) is analogous,
and leads to the contradiction. \( \square \)

**Definition 7.** \( \Gamma \models_\cap (\mathcal{C})B \) iff all three of the following conditions
are fulfilled:

- there is no valuation \( w \) in \( \mathcal{C} \), such that \( w[\Gamma] \subseteq \omega(\mathcal{C}) \), and \( w(A) \notin \omega(\mathcal{C}) \);
- there is no valuation \( w \) in \( \mathcal{C} \), such that \( w[\Gamma] \subseteq \omega(\mathcal{C}) \cup \xi(\mathcal{C}) \),
  and \( w(A) \notin \omega(\mathcal{C}) \cup \xi(\mathcal{C}) \);
- there is no valuation \( w \) in \( \mathcal{C} \), such that \( w[\Gamma] \subseteq \omega(\mathcal{C}) \cup \xi'(\mathcal{C}) \),
  and \( w(A) \notin \omega(\mathcal{C}) \cup \xi'(\mathcal{C}) \).

**Lemma 2.** \( \Gamma \models_\cap (\mathcal{C})B \) iff \( \Gamma \models (\mathfrak{A})B \) and \( \Gamma \models (\mathfrak{B})B \).

**Proof.** (i) Let \( \Gamma \models_\cap (\mathcal{C})B \), and \( \Gamma \models (\mathfrak{A})B \), and \( \Gamma \models (\mathfrak{B})B \). The
reasoning is analogous to the one in Lemma 1.

(ii) Let \( \Gamma \models_\cap (\mathcal{C})B \), and either \( \Gamma \models (\mathfrak{A})B \) or \( \Gamma \models (\mathfrak{B})B \).
Suppose \( \Gamma \models (\mathfrak{A})B \) and \( \Gamma \models (\mathfrak{B})B \). Then there is a valuation \( v^* \) in \( \mathfrak{A} \),
such that \( v^*[\Gamma] \subseteq D(\mathfrak{A}) \) and \( v^*(A) \notin D(\mathfrak{A}) \). Now we have to consider
two possibilities.

(ii.1) There is a valuation \( u^* \) in \( \mathfrak{B} \), such that \( u^*[\Gamma] \subseteq D(\mathfrak{B}) \) and
\( u^*(A) \in D(\mathfrak{B}) \). In this case, from \( v^* \) and \( u^* \) we can obtain a
corresponding valuation \( v^* \) in \( \mathcal{C} \) (see Lemma 1), such that \( v^*[\Gamma] \subseteq \omega(\mathcal{C}) \),
and \( v^*(A) \in \xi'(\mathcal{C}) \). But then \( \Gamma \models_\cap (\mathcal{C})B \), which contradicts
our assumption.

(ii.2) For every valuation \( u \) in \( \mathfrak{B} \), \( u[\Gamma] \notin D(\mathfrak{B}) \). Let \( u' \) be such a
valuation that \( u'[\Gamma] \notin D(\mathfrak{B}) \), and \( u'(A) \notin D(\mathfrak{B}) \). The corresponding
valuation \( w' \) in \( \mathcal{C} \) obtained from \( v^* \) and \( u' \) in the same way as in
Lemma 1 will be such that \( w'[\Gamma] \subseteq \xi(\mathcal{C}) \), and \( w'(A) \in \phi(\mathcal{C}) \). Let \( u'' \)
be such a valuation that \( u''[\Gamma] \notin D(\mathfrak{B}) \), and \( u''(A) \in D(\mathfrak{B}) \). The corresponding
valuation \( w'' \) in \( \mathcal{C} \) obtained from \( v^* \) and \( u'' \) will be such that \( w''[\Gamma] \subseteq \xi(\mathcal{C}) \),
and \( w''(A) \in \xi'(\mathcal{C}) \). Both cases lead us to the contradiction with the assumption that \( \Gamma \models_\cap (\mathcal{C})B \).
The reasoning for $\Gamma \models (A)B$ and $\Gamma \not\models (B)B$ is analogous.

(iii) Let $\Gamma \not\models (C)B$, and $\Gamma \models (A)B$, and $\Gamma \models (B)B$. If $\Gamma \not\models (C)B$, three cases are possible:

(iii.1) There is a valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C)$, and $w(A) \notin \omega(C)$;

(iii.2) There is a valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C) \cup \xi(C)$, and $w(A) \notin \omega(C) \cup \xi(C)$;

(iii.3) There is a valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C) \cup \xi'(C)$, and $w(A) \notin \omega(C) \cup \xi'(C)$.

The reasoning for all three cases is the same. We obtain from $w$ the corresponding valuations $v$ in $A$ and $u$ in $B$ in the same way as we did in Lemma 1. Due to the properties of $w$ described in (iii.1)–(iii.3), either $v$, or $u$, or both of them will be such that they will lead to the contradiction with the assumption that $\Gamma \models (A)B$ and $\Gamma \models (B)B$.

From Lemma 1 we have that $C(C,\models\cup) = C(A) \cup C(B)$. From Lemma 2 we have that $C(C,\models\cap) = C(A) \cap C(B)$. Also, we have that $C(A) = C(B)$ iff $C(A) \cup C(B) = C(A) \cap C(B)$. Therefore, $C(A) = C(B)$ iff $C(C,\models\cup) = C(C,\models\cap)$.

Now let us consider another consequence relation.

**Definition 8.** $\Gamma \models^* (C)B$ iff either

- there is no valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C)$, and $w(A) \notin \omega(C)$,
- and there is no valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C) \cup \xi(C)$, and $w(A) \notin \omega(C) \cup \xi(C)$,
- and there is no valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C) \cup \xi'(C)$, and $w(A) \notin \omega(C) \cup \xi'(C)$,
- or there is a valuation $w$ in $C$, such that $w[\Gamma] \subseteq \omega(C)$, and $w(A) \in \phi(C)$.

**Lemma 3.** $C(C,\models\cup) = C(C,\models\cap)$ iff $\Gamma \models^* (C)B$ for each set of formulas $\Gamma$ and each formula $B$. 
Proof. If $C(\xi, \models \cup) = C(\xi, \models \cap)$, for each $\Gamma$ and $B$ it is true that either $\Gamma \models \cap (C)B$ or $\Gamma \not\models \cup (C)B$. Both cases lead to $\Gamma \models^* (C)B$. Now let us assume that $\Gamma \models^* (C)B$ for some arbitrary $\Gamma$ and $B$. Then (i) for every valuation $w$ in $\xi$, if $w[\Gamma] \subseteq \omega(\xi)$ then $w(A) \in \omega(\xi)$, if $w[\Gamma] \subseteq \omega(\xi) \cup \xi'(!\xi)$, then $w(A) \in \omega(\xi) \cup \xi'(!\xi)$, if $w[\Gamma] \subseteq \omega(\xi) \cap \xi'(!\xi)$, then $w(A) \in \omega(\xi) \cap \xi'(!\xi)$, or (ii) there is at least one valuation in $\xi$, such that all formulas from $\Gamma$ assume a truth value from $\omega(\xi)$, and $B$ assumes a value from $\phi(\xi)$. In the first case $\Gamma \models \cap (C)B$. In the second case $\Gamma \not\models \cup (C)B$. Therefore $C(\xi, \models \cup) = C(\xi, \models \cap)$. □

Below, the number of formulas that need to be considered will be narrowed down to a finite set. I will use the method proposed by J. Kalicki in [1] with necessary modifications.

Lemma 4. For each matrix $\xi_m$, where $m$ is the number of the elements of $V(\xi)$, the following is true: if for each pair $\Gamma$ and $B$ that contains $i \leq m$ different variables $\Gamma \models^* (\xi_m)B$, then for each pair $\Delta$ and $E$ that contains $m + t(t = 0, 1, \cdots)$ different variables $\Delta \models^* (\xi_m)E$.

Proof. Let us use the induction by $t$. For $t = 0$ it is obvious that for each $\Gamma$ and $B$ that contains $i \leq m$ different variables $\Gamma \models^* (\xi_m)B$, then for each pair $\Delta$ and $E$ that contains $m$ different variables $\Delta \models^* (\xi_m)E$.

Let us assume that the theorem is true for $t \leq k$ and prove it for $t = k + 1$. Let there exist $\Delta$ and $E$ that contain $m + k + 1$ different variables, and $\Delta \not\models^* (\xi_m)E$. Then there exists a valuation $w_0$ in $\xi_m$ that maps the variables $p_1, p_2, \cdots, p_{m+k+1}$ to values $x_1, x_2, \cdots, x_{m+k+1}$ respectively, such that either (i) $w_0[\Delta] \subseteq \omega(\xi)$, and $w_0(E) \not\in \omega(\xi)$, or (ii) $w_0[\Delta] \subseteq \omega(\xi) \cup \xi(\xi)$, and $w(E) \not\in \omega(\xi) \cup \xi(\xi)$, or (iii) $w_0[\Delta] \subseteq \omega(\xi) \cap \xi(\xi)$, and $w(E) \not\in \omega(\xi) \cap \xi(\xi)$.

Let us consider (i). Due to the fact that in $\xi_m$ there is $m$ different truth-values in total, there will be at least two $i_1 \neq i_2$ among $i_1 = 1, 2, \cdots, m + k + 1$, such that $x_{i_1} = x_{i_2}$. Now let us consider $\Delta'$ and $E'$, obtained from $\Delta$ and $D$ by replacement of all instances of $p_{i_2}$ with $p_{i_1}$. It is clear that $w_0[\Delta'] \subseteq \omega(\xi_m)$ and $w_0(E') \not\in \omega(\xi_m)$. Because $\Delta'$ and $E'$ contain $m + k$ different variables, according to the inductive assumption, $\Delta' \models^* (\xi_m)E'$. Therefore, there exists a valuation $w^*$ in
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\( \mathcal{C}_m \), which maps the variables \( p_1, p_2, \ldots, p_{i_2-1}, p_{i_2+1}, \ldots, p_{m+k+1} \) on the values \( y_1, y_2, \ldots, y_{i_2-1}, y_{i_2+1}, \ldots, y_{m+k+1} \) respectively, such that

\[ w^*[\Delta] \subseteq \omega(\mathcal{C}_m) \text{ and } w^*(E') \in \phi(\mathcal{C}_m). \]

In this case we can construct a valuation \( w^* \), which maps the variables \( p_1, p_2, \ldots, p_{m+k+1} \) on the values \( y_1, y_2, \ldots, y_{i_2-1}, y_{i_2+1}, \ldots, y_{m+k+1} \) respectively, such that

\[ w^* \subseteq \omega(\mathcal{C}_m) \text{ and } w^*(E) \in \phi(\mathcal{C}_m). \]

In this case we can construct a valuation \( w^* \), which maps the variables \( p_1, p_2, \ldots, p_{m+k+1} \) on the values \( y_1, y_2, \ldots, y_{i_2-1}, y_{i_2+1}, \ldots, y_{m+k+1} \) respectively, such that

\[ w^*[\Delta] \subseteq \omega(\mathcal{C}_m) \text{ and } w^*(E) \in \phi(\mathcal{C}_m). \]

But then \( \Delta \models^* (\mathcal{C}_m)E \), which contradicts our assumption.

The reasoning for (ii) and (iii) is analogous.

For \( m \) different variables there is \( k = m^m \) different valuations \( v_1, v_2, \ldots, v_k \) in \( \mathcal{C}_m \). We can assign to each variable \( p_i (1 \leq i \leq m) \) a unique value-sequence \( |p_i| = < x_1, x_2, \ldots, x_k > \), where \( x_l = v_l(p_i) \) \((1 \leq l \leq k)\).

Now let us construct the following sequence of the classes of formulas:

- The elements of \( \mathcal{C}_0 \) are the variables \( p_1, p_2, \ldots, p_m \) exclusively;

- to a class \( \mathcal{C}_{t+1} \) belong all formulas that can be constructed by means of one connective, an element of class \( \mathcal{C}_t \), and (if needed) elements of \( \mathcal{C}_{n \leq t} \).

For each formula \( B \) from \( \mathcal{C}_n \) we can calculate the corresponding value-sequence \( |B| = < y_1, y_2, \ldots, y_k > \), where \( y_j (1 \leq j \leq k) \) is obtained from \( j \)-th elements of sequences assigned to the variables included in \( B \). Let us denote the set of value-sequences for elements of \( \mathcal{C}_n \) as \( |\mathcal{C}_n| \). Because the sequences in question consist of \( k \) elements, and the number of truth-values equals \( m \), in total there is \( m^k \) possible sequences. Therefore, there is a finite \( n_0 \leq m^k \), such that \( |\mathcal{C}_{n_0}| \) contains no value-sequence which is not also the element of some \( |\mathcal{C}_{n < n_0}| \).

**Lemma 5.** The value-sequence of any formula \( B \in \mathcal{C}_{n > n_0} \) is identical to some element of \( |\mathcal{C}_{n < n_0}| \).

**Proof.** Let \( B \in \mathcal{C}_{n_0+1} \). By definition of \( \mathcal{C}_{n_0+1} \), formula \( B \) consists of the main connective, at least one formula from \( \mathcal{C}_{n_0} \),
and probably elements of $CL_{n_i<n_0}$. By definition of $n_0$, each value-sequence from $|CL_{n_0}|$ is also present in some $|CL_{n_i<n_0}|$. Therefore, by definition of $|CL|$, there is a set $|CL_{\text{max}(i,j)+1}|$, which contains the value-sequence identical to $|B|$. Because $n_i < n_0$ and $n_j < n_0$, we have that $\text{max}(n_i, n_j) + 1 \leq n_0$, $|B| \in |CL_{n_i<n_0}|$. From that, according to the definition of $n_0$, we obtain that $|B| \in |CL_{n<n_0}|$. The theorem is proved for $CL_{n_0+1}$. The generalization for $CL_{n>n_0}$ is obvious.

So the set $|CL_1| \cup |CL_2| \cup \cdots \cup |CL_{n_0}|$ contains all value-sequences possible in $C_m$ for formulas that contain no more than $m$ different variables. From this fact and Lemma 4 it follows that $\Gamma \models^* (C_m)B$ for each $\Gamma$ and $B$ iff $\Delta \models^* (C_m)E$ for every $\Delta$ and $E$ that consist exclusively of the elements of $CL_1 \cup CL_2 \cup \cdots \cup CL_{n_0}$.

This concludes the construction of the procedure for testing if $C(\mathfrak{A}) = C(\mathfrak{B})$ for two arbitrary finite-valued matrices $\mathfrak{A}$ and $\mathfrak{B}$ for some propositional language $L$.

References
