Correspondence analysis for strong three-valued logic

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Abstract. I apply Kooi and Tamminga’s (2012) idea of correspondence analysis for many-valued logics to strong three-valued logic ($K_3$). First, I characterize each possible single entry in the truth-table of a unary or a binary truth-functional operator that could be added to $K_3$ by a basic inference scheme. Second, I define a class of natural deduction systems on the basis of these characterizing basic inference schemes and a natural deduction system for $K_3$. Third, I show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics. Among other things, I thus obtain a new proof system for Łukasiewicz’s three-valued logic.

Keywords: three-valued logic, correspondence analysis, proof theory, natural deduction systems

1 Introduction

Strong three-valued logic ($K_3$) [1] and Łukasiewicz’s three-valued logic $L_3$ [2] have much in common: their truth-tables for negation, disjunction, and conjunction coincide, and they have the same concept of validity. The two logics differ, however, in their treatment of implication: whereas Kleene’s implication is definable in terms of negation, disjunction, and conjunction, this does not hold true for Łukasiewicz’s implication ($L_3$ is therefore a truth-functional extension of $K_3$). This fact seriously complicates the construction of proof systems for $L_3$.

In this paper, I present a general method for finding natural deduction systems for truth-functional extensions of $K_3$. To do so, I use the correspondence analysis for many-valued logics that was
presented recently by [3]. In their study of the logic of paradox (LP) [4], they characterize every possible single entry in the truth-table of a unary or a binary truth-functional operator by a basic inference scheme. As a consequence, each unary and each binary truth-functional operator is characterized by a set of basic inference schemes. Kooi and Tamminga show that if we add the inference schemes that characterize an operator to a natural deduction system for LP, we immediately obtain a natural deduction system that is sound and complete with respect to the logic that contains, next to LP’s negation, disjunction, and conjunction, the additional operator. In this paper, I show that the same thing can be done for K₃.

The structure of my paper is as follows. First, I briefly present K₃. Second, I give a list of basic inference schemes that characterize every possible single entry in the truth-table of a unary or a binary truth-functional operator. Third, I define a class of natural deduction systems on the basis of these characterizing inference schemes and a natural deduction system for K₃. I show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics.

2 Strong three-valued logic (K₃)

Strong three-valued logic (K₃) provides an alternative way to evaluate formulas from a propositional language L built from a set P = {p, p’, . . .} of atomic formulas using negation (¬), disjunction (∨), and conjunction (∧). K₃ adds a third truth-value ‘none’ to the classical pair ‘false’ and ‘true’. In K₃, a valuation is a function v from the set P of atomic formulas to the set {0, i, 1} of truth-values ‘false’, ‘none’, and ‘true’. A valuation v on P is extended recursively to a valuation on L by the following truth-tables for ¬, ∨, and ∧:

\[
\begin{array}{c|c}
\hline
f_\neg & 0 & 1 \\
\hline
0 & 1 & 0 \\
1 & i & i \\
i & i & i \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
f_\lor & 0 & i & 1 \\
\hline
0 & 0 & 0 & 0 \\
i & i & i & i \\
1 & 1 & 1 & 1 \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
f_\land & 0 & i & 1 \\
\hline
0 & 0 & 0 \\
i & i & i \\
1 & i & 1 \\
\hline
\end{array}
\]
An argument from a set $\Pi$ of premises to a conclusion $\phi$ is valid (notation: $\Pi \models \phi$) if and only if for each valuation $v$ it holds that if $v(\psi) = 1$ for all $\psi$ in $\Pi$, then $v(\phi) = 1$.

3 Correspondence Analysis for $K_3$

Let $L(\sim)_m(\circ)_n$ be the language built from the set $P = \{ p, p', \ldots \}$ of atomic formulas using negation ($\sim$), disjunction ($\lor$), conjunction ($\land$), $m$ unary operators $\sim_1, \ldots, \sim_m$, and $n$ binary operators $\circ_1, \ldots, \circ_n$. It is obvious that $L(\sim)_m(\circ)_n$ is an extension of $L$. To interpret this extended language, I use $K_3$’s concept of validity, the truth-tables $f_\sim$, $f_\lor$, and $f_\land$, but also the truth-tables $f_{\sim_1}, \ldots, f_{\sim_m}$ and the truth-tables $f_{\circ_1}, \ldots, f_{\circ_n}$. I refer to the resulting logic as $K_3(\sim)_m(\circ)_n$.

To construct a proof system for $K_3(\sim)_m(\circ)_n$, I follow [3]. I first characterize each possible single entry in the truth-table of a unary or a binary operator by a basic inference scheme. To do so, I need the following notion of single entry correspondence [3, p. 722]:

**Definition 1 (Single Entry Correspondence).** Let $\Pi \subseteq L(\sim)_m(\circ)_n$ and let $\phi \in L(\sim)_m(\circ)_n$. Let $x, y, z \in \{0, i, 1\}$. Let $E$ be a truth-table entry of the type $f_\sim(x) = y$ or $f_\circ(x, y) = z$. Then the truth-table entry $E$ is characterized by an inference scheme $\Pi/\phi$, if

$$ E \text{ if and only if } \Pi \models \phi. $$

Accordingly, each of the nine possible single entries in a truth-table $f_\sim$ for a unary operator $\sim$ and each of the twenty-seven possible entries in a truth-table $f_\circ$ for binary operator $\circ$ is characterized by an inference scheme (I do the binary operator case first):

**Theorem 1.** Let $\phi, \psi, \chi \in L(\sim)_m(\circ)_n$. Then

$$ f_\circ(0, 0) = \begin{cases} 
0 & \text{iff } \neg \phi \land \neg \psi \models \neg(\phi \circ \psi) \\
i & \text{iff } \neg \phi \land \neg \psi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \chi \\
1 & \text{iff } \neg \phi \land \neg \psi \models \phi \circ \psi
\end{cases} $$
\[ f_0(0, i) = \begin{cases} 
0 & \text{iff } \neg \phi \models (\psi \lor \neg \psi) \lor \neg(\phi \circ \psi) \\
i & \text{iff } \neg \phi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \psi \lor \neg \psi \\
1 & \text{iff } \neg \phi \models (\psi \lor \neg \psi) \lor (\phi \circ \psi) 
\end{cases} \]

\[ f_0(0, 1) = \begin{cases} 
0 & \text{iff } \neg \phi \land \psi \models \neg(\phi \circ \psi) \\
i & \text{iff } \neg \phi \land \psi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \chi \\
1 & \text{iff } \neg \phi \land \psi \models \phi \circ \psi 
\end{cases} \]

\[ f_0(i, 0) = \begin{cases} 
0 & \text{iff } \models (\phi \lor \neg \phi) \lor (\psi \lor \neg \psi) \lor \neg(\phi \circ \psi) \\
i & \text{iff } (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models (\phi \lor \neg \phi) \lor (\psi \lor \neg \psi) \\
1 & \text{iff } \models (\phi \lor \neg \phi) \lor (\psi \lor \neg \psi) \lor (\phi \circ \psi) 
\end{cases} \]

\[ f_0(i, i) = \begin{cases} 
0 & \text{iff } \psi \models (\phi \lor \neg \phi) \lor \neg(\phi \circ \psi) \\
i & \text{iff } \psi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \phi \lor \neg \phi \\
1 & \text{iff } \psi \models (\phi \lor \neg \phi) \lor (\phi \circ \psi) 
\end{cases} \]

\[ f_0(1, 0) = \begin{cases} 
0 & \text{iff } \phi \land \neg \psi \models \neg(\phi \circ \psi) \\
i & \text{iff } \phi \land \neg \psi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \chi \\
1 & \text{iff } \phi \land \neg \psi \models \phi \circ \psi 
\end{cases} \]

\[ f_0(1, i) = \begin{cases} 
0 & \text{iff } \phi \models (\psi \lor \neg \psi) \lor \neg(\phi \circ \psi) \\
i & \text{iff } \phi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \psi \lor \neg \psi \\
1 & \text{iff } \phi \models (\psi \lor \neg \psi) \lor (\phi \circ \psi) 
\end{cases} \]

\[ f_0(1, 1) = \begin{cases} 
0 & \text{iff } \phi \land \psi \models \neg(\phi \circ \psi) \\
i & \text{iff } \phi \land \psi, (\phi \circ \psi) \lor \neg(\phi \circ \psi) \models \chi \\
1 & \text{iff } \phi \land \psi \models \phi \circ \psi 
\end{cases} \]

**Proof.** Case \( f_0(0, 0) = 0 \). (\Rightarrow\) Suppose that \( \neg \phi \land \neg \psi \not\models \neg(\phi \circ \psi) \). Then there is a valuation \( v \) such that \( v(\neg \phi \land \neg \psi) = 1 \) and \( v(\neg(\phi \circ \psi)) = 0 \) or \( v(\phi \circ \psi) = 0 \) for some \( i \). In either case, one of the conditions for \( f_0(i, 0) = 0 \) is satisfied. Therefore, \( f_0(i, 0) = 1 \) or \( f_0(i, 0) = 0 \). If \( f_0(i, 0) = 0 \), then \( f_0(i, 0) = 0 \). If \( f_0(i, 0) = 1 \), then \( f_0(i, 0) = 1 \). In either case, \( f_0(i, 0) = 0 \) or \( f_0(i, 0) = 1 \). Therefore, \( f_0(0, 0) = 0 \).
ψ)) \neq 1. Then v(ϕ) = 0, v(ψ) = 0, and v(ϕ \circ ψ) \neq 0. Therefore, it must be that f_{\circ}(0, 0) \neq 0.

(⇐) Suppose that ϕ, (ϕ \circ ψ) \lor ¬(ϕ \circ ψ) \neq ψ \lor ¬ψ.
Then there is a valuation v such that v(ϕ) = 1, v((ϕ \circ ψ) \lor ¬(ϕ \circ ψ)) = 1 and v(ψ \lor ¬ψ) \neq 1. Then v(ϕ) = 1, v(ψ) = i, and v(ϕ \circ ψ) \neq i.
Therefore, it must be that f_{\circ}(1, i) \neq i.

(⇒) Suppose that ϕ, (ϕ \circ ψ) \lor ¬(ϕ \circ ψ) \neq ψ \lor ¬ψ.
Then there is a valuation v such that v(ϕ) = 1, v((ϕ \circ ψ) \lor ¬(ϕ \circ ψ)) = 1 and v(ψ \lor ¬ψ) \neq 1. Then v(ϕ) = 1, v(ψ) = i, and v(ϕ \circ ψ) \neq i.
Therefore, it must be that f_{\circ}(1, i) = i.

The other cases are proved similarly.

\textbf{Theorem 2.} Let ϕ, ψ ∈ L_{(\sim)^m(o)n}. Then

\begin{align*}
f_{\sim}(0) &= \begin{cases} 
0 & \text{iff } \neg \phi \models \neg \sim \phi \\
i & \text{iff } \neg \phi, (\sim \phi \lor \sim \phi) \models \psi \\
i & \text{iff } \neg \phi \models \sim \phi
\end{cases} \\
f_{\sim}(i) &= \begin{cases} 
0 & \text{iff } \models (\phi \lor \neg \phi) \lor \neg \sim \phi \\
i & \text{iff } (\sim \phi \lor \sim \phi) \models \phi \lor \neg \phi \\
i & \text{iff } \models (\phi \lor \neg \phi) \lor \sim \phi
\end{cases} \\
f_{\sim}(1) &= \begin{cases} 
0 & \text{iff } \phi \models \sim \phi \\
i & \text{iff } \phi, (\sim \phi \lor \sim \phi) \models \psi \\
i & \text{iff } \phi \models \sim \phi
\end{cases}
\end{align*}

\textbf{Proof.} Adapt the proof of the previous theorem.

As a result, given K_{3}'s concept of validity and its truth-tables f_{\sim}, f_{\lor}, and f_{\land}, each unary operator \sim_k (1 ≤ k ≤ m) is characterized by
the set of three basic inference schemes that characterize the three single entries in its truth-table \( f_{\sim_k} \) and each binary operator \( \circ_l \) \((1 \leq l \leq n)\) is characterized by the set of nine basic inference schemes that characterize the nine single entries in its truth-table \( f_{\circ_l} \). The inference schemes that characterize a truth-table are independent.

4 Natural deduction systems

I now use the characterizations of the previous section to construct proof systems for truth-functional extensions of \( K_3 \). First, I define a natural deduction system \( \text{ND}_{K_3} \) which I later show to be sound and complete with respect to \( K_3 \) (this is a corollary of my main theorem). Second, on the basis of \( \text{ND}_{K_3} \) and Theorems 1 and 2, I define a natural deduction system for the logic \( K_3(\sim)_m(\circ)_n \) as follows: for each unary operator \( \sim_k \) \((1 \leq k \leq m)\) I add its three characterizing basic inference schemes as derivation rules to \( \text{ND}_{K_3} \) and for each binary operator \( \circ_l \) \((1 \leq l \leq n)\) I add its nine characterizing inference schemes as derivation rules to \( \text{ND}_{K_3} \). Third, I show, using a Henkin-style proof, that the resulting natural deduction system is sound and complete with respect to the logic \( K_3(\sim)_m(\circ)_n \).

My proof-theoretical study of \( K_3 \) closely follows Kooi and Tamminga’s (2012) proof-theoretical study of \( LP \). In fact, to construct natural deduction systems for extensions of \( K_3 \) and to prove their soundness and completeness, I only slightly adapt Kooi and Tamminga’s definitions, lemmas and theorems on extensions of \( LP \).

Let me first define the natural deduction system \( \text{ND}_{K_3} \).¹

**Definition 2.** Derivations in the system \( \text{ND}_{K_3} \) are inductively defined as follows:

**Basis:** The proof tree with a single occurrence of an assumption \( \phi \) is a derivation.

**Induction Step:** Let \( D, D_1, D_2, D_3 \) be derivations. Then they can be extended by the following rules (double lines indicate that the rules work both ways):

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¹For the notational conventions, see [5].
On the basis of $\text{ND}_{K_3}$, I now define a natural deduction system for the logic $K_3(\sim)_m(\circ)_n$. The Theorems 1 and 2 tell me that each truth-table $f_{\sim_k}$ is characterized by three basic inference schemes and that each truth-table $f_{\circ_l}$ is characterized by nine basic inference schemes. I obtain a new natural deduction system for the logic $K_3(\sim)_m(\circ)_n$ by adding to $\text{ND}_{K_3}$ these characterizing basic inference schemes as derivation rules.

More specifically, for each basic inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\sim_k}(x) = y$ in the truth-table $f_{\sim_k}$, I add the derivation rule

$$
\frac{\psi_1 \cdots \psi_j}{\phi} \quad R_{\sim_k}(x, y)
$$

to the natural deduction system $\text{ND}_{K_3}$. Similarly, for each basic inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\circ_l}(x, y) = z$ in the truth-table $f_{\circ_l}$, I add the derivation rule

$$
\frac{\psi_1 \cdots \psi_j}{\phi} \quad R_{\circ_l}(x, y, z)
$$

to the natural deduction system $\text{ND}_{K_3}$. 

\[
\frac{D_1 \quad D_2}{\psi \quad \neg\phi \quad E\Phi}
\]

\[
\frac{D_1 \quad D_2}{\phi \quad \psi \quad \land I} \quad \frac{D}{\phi \quad \land \psi \quad \land E_1} \quad \frac{D}{\phi \quad \land \psi \quad \land E_2}
\]

\[
\frac{D}{\phi \quad \lor I_1} \quad \frac{D}{\psi \quad \lor I_2} \quad \frac{D_1}{[\phi]^u} \quad \frac{D_2}{[\psi]^v} \quad \frac{D_3}{\chi} \quad \frac{\chi \quad \land E^{u, v}}{
\frac{D_1 \quad D_2 \quad D_3}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I} \quad \frac{D}{\phi \quad \land I}
\]

\[
\frac{D_1 \quad D_2 \quad D_3}{\phi \quad \neg\phi \quad D\land \neg\phi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi} \quad \frac{D}{\neg\phi \quad \land \neg\psi}
\]

\[
\frac{D_1 \quad D_2 \quad D_3}{[\phi]^u \quad [\psi]^v \quad \chi \quad \land E^{u, v}}
\]
For instance, assume that $f_0(0,0) = 0$ is one of the truth-table entries in $f$. Then, because Theorem 1 tells me that $f_0(0,0) = 0$ is characterized by the basic inference scheme $\neg \phi \land \neg \psi \vdash \neg (\phi \circ \psi)$, I add the derivation rule

$$\frac{\neg \phi \land \neg \psi}{\neg (\phi \circ \psi)}$$

R_0(0,0,0)

to the natural deduction system ND$_{K_3}$.

In this way, I define the system ND$_{K_3} \cup \bigcup_{k=1}^m \{R_{\sim_k}(x,y) : f_{\sim_k}(x) = y\} \cup \bigcup_{l=1}^n \{R_{\circ_l}(x,y,z) : f_{\circ_l}(x,y) = z\}$, which I refer to as ND$_{K_3(\sim)(\circ)n}$. I now show that this natural deduction system is sound and complete with respect to the logic $K_3(\sim)(\circ)n$.

### 4.1 Soundness of ND$_{K_3(\sim)(\circ)n}$

A conclusion $\phi$ is derivable from a set $\Pi$ of premises (notation: $\Pi \vdash \phi$) if and only if there is a derivation in the system ND$_{K_3(\sim)(\circ)n}$ of $\phi$ from $\Pi$.

The system’s local soundness is easy to establish:

**Lemma 1 (Local Soundness).** Let $\Pi, \Pi', \Pi'' \subseteq L(\sim)(\circ)n$ and let $\phi, \psi \in L(\sim)(\circ)n$. Then

(i) If $\phi \in \Pi$, then $\Pi \models \phi$

(ii) If $\Pi \models \phi$ and $\Pi' \models \neg \phi$, then $\Pi, \Pi' \models \phi$

(iii) If $\Pi \models \phi$ and $\Pi' \models \psi$, then $\Pi, \Pi' \models \phi \land \psi$

(iv) If $\Pi \models \phi \land \psi$, then $\Pi \models \phi$

(v) If $\Pi \models \phi \land \psi$, then $\Pi \models \psi$

(vi) If $\Pi \models \phi$, then $\Pi \models \phi \lor \psi$

(vii) If $\Pi \models \psi$, then $\Pi \models \phi \lor \psi$

(viii) If $\Pi \models \phi \lor \psi$ and $\Pi', \phi \models \chi$ and $\Pi'', \psi \models \chi$, then $\Pi, \Pi', \Pi'' \models \chi$

(ix) $\Pi \models \phi$ if and only if $\Pi \models \neg \neg \phi$

(x) $\Pi \models \neg (\phi \lor \psi)$ if and only if $\Pi \models \neg \phi \land \neg \psi$

(xi) $\Pi \models \neg (\phi \land \psi)$ if and only if $\Pi \models \neg \phi \lor \neg \psi$.

**Theorem 3 (Soundness).** Let $\Pi \subseteq L(\sim)(\circ)n$ and let $\phi \in L(\sim)(\circ)n$. Then

If $\Pi \vdash \phi$, then $\Pi \models \phi$. 
Proof. By induction on the depth of derivations. The local soundness of the rules of the basic natural deduction system $\text{ND}_K^3$ follows from the previous lemma. For each unary operator $\sim_k$ ($1 \leq k \leq m$) the local soundness of the three derivation rules in $\{R_{\sim_k}(x,y): f_{\sim_k}(x) = y\}$ follows from Theorem 2. For each binary operator $\circ_l$ ($1 \leq l \leq n$) the local soundness of the nine derivation rules in $\{R_{\circ_l}(x,y,z): f_{\circ_l}(x,y) = z\}$ follows from Theorem 1.

4.2 Completeness of $\text{ND}_{K_3}^{(\sim)}$

In my completeness proof, consistent prime theories are the syntactical counterparts of valuations:

Definition 3. Let $\Pi \subseteq L^{(\sim)}$. Then $\Pi$ is a consistent prime theory (CPT), if

(i) $\Pi \neq L^{(\sim)}$ (consistency)
(ii) If $\Pi \vdash \phi$, then $\phi \in \Pi$ (closure)
(iii) If $\phi \lor \psi \in \Pi$, then $\phi \in \Pi$ or $\psi \in \Pi$ (primeness).

The syntactical counterpart of the truth-value of a formula under a valuation is a formula’s elementhood in a consistent prime theory:

Definition 4. Let $\Pi \subseteq L^{(\sim)}$ and let $\phi \in L^{(\sim)}$. Then $\phi$’s elementhood in $\Pi$ (notation: $e(\phi, \Pi)$) is defined as follows:

$$e(\phi, \Pi) = \begin{cases} 
\emptyset, & \text{if } \phi \in \Pi \text{ and } \neg \phi \notin \Pi \\
0, & \text{if } \phi \notin \Pi \text{ and } \neg \phi \in \Pi \\
i, & \text{if } \phi \notin \Pi \text{ and } \neg \phi \notin \Pi \\
1, & \text{if } \phi \in \Pi \text{ and } \neg \phi \notin \Pi.
\end{cases}$$

To ensure that in the presence of an operator the notion of elementhood behaves in conformity with the operator’s truth-tables, I need the following lemma:

Lemma 2. Let $\Pi$ be a CPT and let $\phi, \psi \in L^{(\sim)}$. Then

(i) $e(\phi, \Pi) \neq \emptyset$
(ii) $f_{\sim}(e(\phi, \Pi)) = e(\neg \phi, \Pi)$
(iii) $f_{\lor}(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \lor \psi, \Pi)$
(iv) $f_{\land}(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \land \psi, \Pi)$
(v) $f_{\sim_k}(e(\phi, \Pi)) = e(\sim_k \phi, \Pi)$ for $1 \leq k \leq m$
(vi) $f_{\circ_l}(e(\phi, \Pi), e(\psi, \Pi)) = e(\phi \circ_l \psi, \Pi)$ for $1 \leq l \leq n$. 
Proof.

(i) Suppose $e(\phi, \Pi) = \emptyset$. Then $\phi \in \Pi$ and $\neg \phi \in \Pi$. Then $\Pi \vdash \phi$ and $\Pi \vdash \neg \phi$. By the rule $EFQ$, it must be that $\Pi \vdash \psi$ for all $\psi \in L_{(\sim)_m(\circ)_n}$. By closure, $\psi \in \Pi$ for all $\psi \in L_{(\sim)_m(\circ)_n}$. Then $\Pi = L_{(\sim)_m(\circ)_n}$. Contradiction.

(ii) Suppose $e(\phi, \Pi) = 0$. Then $\phi \notin \Pi$ and $\neg \phi \in \Pi$. By closure and the rule $DN$, $\neg \phi \in \Pi$ and $\neg \neg \phi \notin \Pi$. Hence, $e(\neg \phi, \Pi) = 1 = f_-(0) = f_-(e(\phi, \Pi))$.

Suppose $e(\phi, \Pi) = i$. Then $\phi \in \Pi$ and $\neg \phi \in \Pi$. By closure and the rule $DN$, $\neg \phi \in \Pi$ and $\neg \neg \phi \in \Pi$. Hence, $e(\neg \phi, \Pi) = i = f_-(i) = f_-(e(\phi, \Pi))$.

Suppose $e(\phi, \Pi) = 1$. Then $\phi \in \Pi$ and $\neg \phi \notin \Pi$. By closure and the rule $DN$, $\neg \phi \notin \Pi$ and $\neg \neg \phi \in \Pi$. Hence, $e(\neg \phi, \Pi) = 0 = f_-(1) = f_-(e(\phi, \Pi))$.

(iii) I prove the cases for (1) $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$, (2) $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$, and (3) $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. The other six cases are proved similarly.

(1) Suppose $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$. Then $\phi \notin \Pi$, $\psi \notin \Pi$, $\neg \phi \in \Pi$, and $\neg \psi \in \Pi$. By primeness, $\phi \lor \psi \notin \Pi$. By closure and the rules $\land I$ and $\lor I_0$, $\neg (\phi \lor \psi) \in \Pi$. Hence, $e(\phi \lor \psi, \Pi) = 0 = f_\lor(0, 0) = f_\lor(e(\phi, \Pi), e(\psi, \Pi))$.

(2) Suppose $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg \phi \in \Pi$, and $\neg \psi \in \Pi$. By closure and the rule $\lor I_1$, $\phi \lor \psi \in \Pi$. By closure and the rules $\land I$ and $\lor I_1$, $\neg (\phi \lor \psi) \in \Pi$. Hence, $e(\phi \lor \psi, \Pi) = i = f_\lor(i, i) = f_\lor(e(\phi, \Pi), e(\psi, \Pi))$.

(3) Suppose $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg \phi \notin \Pi$, and $\neg \psi \in \Pi$. By closure and the rule $\lor I_1$, $\phi \lor \psi \in \Pi$. By closure and the rules $\land E_1$ and $\lor I_0$, $\neg (\phi \lor \psi) \notin \Pi$. Hence, $e(\phi \lor \psi, \Pi) = 1 = f_\lor(1, i) = f_\lor(e(\phi, \Pi), e(\psi, \Pi))$.

(iv) Analogous to (iii).
There are three cases for each \( \sim_k \) (1 \( \leq \) \( k \) \( \leq \) \( n \)). (For readability, the subscript \( k \) is dropped in the remainder of this proof.) I prove the case for \( e(\phi, \Pi) = 0 \). The other two cases are proved similarly.

Suppose \( e(\phi, \Pi) = 0 \). Then \( \phi \not\in \Pi \) and \( \neg \phi \in \Pi \). There are three cases:

1. Suppose \( R_{\sim}(0,0) \) is one of the three rules for \( \sim \) in \( \text{ND}_{K_{3}}(\sim)_m(\circ)_n \). Then \( f_{\sim}(0) = 0 \). By closure and the rule \( R_{\sim}(0,0) \), it must be that \( \neg \sim \phi \in \Pi \). By (i), it must be that \( \sim \phi \not\in \Pi \). Therefore, \( e(\sim \phi, \Pi) = 0 = f_{\sim}(e(\phi, \Pi)) \).

2. Suppose \( R_{\sim}(0,i) \) is one of the three rules for \( \sim \) in \( \text{ND}_{LP}(\sim)_m(\circ)_n \). Then \( f_{\sim}(0) = i \). By closure, the fact that \( \Pi \) is a CPT, and the rule \( R_{\sim}(0,i) \), it must be that \( \sim \phi \lor \neg \sim \phi \not\in \Pi \). By closure and the rules \( \lor I_1 \) and \( \lor I_2 \), \( \sim \phi \not\in \Pi \) and \( \neg \sim \phi \not\in \Pi \). Therefore, \( e(\sim \phi, \Pi) = i = f_{\circ}(0) = f_{\sim}(e(\phi, \Pi)) \).

3. Suppose \( R_{\sim}(0,1) \) is one of the three rules for \( \sim \) in \( \text{ND}_{LP}(\sim)_m(\circ)_n \). Analogous to (1).

(vi) Analogous to (v).
(i) $\Pi \subseteq \Pi^*$
(ii) $\Pi^* \not\vdash \phi$
(iii) $\Pi^*$ is a CPT.

**Proof.** Suppose that $\Pi \not\vdash \phi$. Let $\psi_1, \psi_2, \ldots$ be an enumeration of $L_{L(\neg)m(o)n}$. I define the sequence $\Pi_0, \Pi_1, \ldots$ of sets of formulas as follows:

$$
\begin{align*}
\Pi_0 &= \Pi \\
\Pi_{i+1} &= \begin{cases} 
\Pi_i \cup \{\psi_{i+1}\}, & \text{if } \Pi_i \cup \{\psi_{i+1}\} \not\vdash \phi \\
\Pi_i, & \text{otherwise.}
\end{cases}
\end{align*}
$$

Take $\Pi^* = \bigcup_{n \in \mathbb{N}} \Pi_n$. Standard proofs show that (i), (ii), and (iii) hold. \(\square\)

**Theorem 4 (Completeness).** Let $\Pi \subseteq L_{L(\neg)m(o)n}$ and let $\phi \in L_{L(\neg)m(o)n}$. Then

If $\Pi \models \phi$, then $\Pi \vdash \phi$.

**Proof.** By contraposition. Suppose $\Pi \not\vdash \phi$. By the Lindenbaum lemma, there is a CPT $\Pi^*$ such that $\Pi \subseteq \Pi^*$ and $\Pi^* \not\vdash \phi$. Let $v_{\Pi^*}$ be the valuation introduced in the truth lemma. By the truth lemma, it holds that $v_{\Pi^*}(\psi) = 1$ for all $\psi$ in $\Pi$ and $v_{\Pi^*}(\phi) \neq 1$. Therefore, $\Pi \not\models \phi$. \(\square\)

**Corollary 1.** The system $\text{ND}_{K_3}$ is sound and complete with respect to $K_3$.

**Proof.** Consider the logic $K_3\neg$ that is obtained from $K_3$ by adding $K_3$’s truth-table $f_\neg$ for negation to it. Evidently, $K_3\neg$ is $K_3$. By the soundness and completeness theorems, $\text{ND}_{K_3\neg}$ is sound and complete with respect to $K_3\neg$. It is easy to see that the rules $R_-{(0,1)}, R_-{(i,i)}$, and $R_-{(1,0)}$ are derived rules in $\text{ND}_{K_3}$. \(\square\)
5 Łukasiewicz’s three-valued logic ($L_3$)

Let me illustrate this general method for finding natural deduction systems for truth-functional extensions of $K_3$ with Łukasiewicz’s three-valued logic ($L_3$). $L_3$ evaluates arguments consisting of formulas from a propositional language $L_3$ built from a set $P = \{p, p', \ldots\}$ of atomic formulas using negation ($\neg$), disjunction ($\lor$), conjunction ($\land$), and implication ($\supset$). $L_3$ has the same valuations as $K_3$: in $L_3$, a valuation is a function $v$ from the set $P$ of atomic formulas to the set $\{0, i, 1\}$ of truth-values. A valuation $v$ on $P$ is extended recursively to a valuation on $L_3$ by the truth-tables for $\neg$, $\lor$, and $\land$, and the truth-table for $\supset$:

<table>
<thead>
<tr>
<th>$f_\supset$</th>
<th>0</th>
<th>i</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>i</td>
<td>1</td>
</tr>
</tbody>
</table>

$L_3$ has the same concept of validity as $K_3$: an argument from a set $\Pi$ of premises to a conclusion $\phi$ is valid (notation: $\Pi \models \phi$) if and only if for each valuation $v$ it holds that if $v(\psi) = 1$ for all $\psi$ in $\Pi$, then $v(\phi) = 1$.

Theorem 1 tells me that the truth-table $f_\supset$ is characterized by the following nine basic inference schemes:

- $f_\supset(0, 0) = 1$ iff $\neg\phi \land \neg\psi \models \phi \supset \psi$
- $f_\supset(0, i) = 1$ iff $\neg\phi \models (\psi \lor \neg\psi) \lor \phi \supset \psi$
- $f_\supset(0, 1) = 1$ iff $\neg\phi \land \psi \models \phi \supset \psi$
- $f_\supset(i, 0) = i$ iff $\neg\psi, (\phi \supset \psi) \lor \neg(\phi \supset \psi) \models \phi \lor \neg\phi$
- $f_\supset(i, i) = 1$ iff $\models (\phi \lor \neg\phi) \lor (\psi \lor \neg\psi) \lor (\phi \supset \psi)$
- $f_\supset(i, 1) = 1$ iff $\psi \models (\phi \lor \neg\phi) \lor (\phi \supset \psi)$
- $f_\supset(1, 0) = 0$ iff $\phi \land \neg\psi \models \neg(\phi \supset \psi)$
- $f_\supset(1, i) = i$ iff $\phi, (\phi \supset \psi) \lor \neg(\phi \supset \psi) \models \psi \lor \neg\psi$
- $f_\supset(1, 1) = 1$ iff $\phi \land \psi \models \phi \supset \psi$.

From Theorems 3 and 4 it follows that the natural deduction system $\text{ND}_{K_3 \supset}$, obtained from adding these nine basic inference schemes as derivation rules to the natural deduction system $\text{ND}_{K_3}$, is sound and complete with respect to $L_3$. The general method I
presented in this paper, therefore, makes it easy to find natural deduction systems for truth-functional extensions of $K_3$.

6 Conclusion

Next to Kooi and Tamminga’s (2012) proof-theoretical study of $LP$, the present investigation of $K_3$ is only a second step in the study of many-valued logics using correspondence analysis. At the current stage of research, the following questions seem pressing. Which many-valued logics can be studied using correspondence analysis? Which many-valued logics cannot? Are there some characteristics a many-valued logic must have to be amenable to correspondence analysis?

References


