Non-classical Logic
Неклассическая логика

ALEXANDER A. BELIKOV

Vojshvillo-Style Semantics for Some Extensions of FDE: Part I

Dedicated to
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In this paper I examine the semantics of semi-generalized state descriptions — a kind of the informational semantics for logic of first-degree entailments (FDE) proposed by E. K. Vojshvillo in the early eighties. A key feature of the approach is to consider state descriptions, which do not satisfy the classic ontological conditions of consistency and completeness that allows to determine a relevant entailment. By relevant entailment we understand such a relation, that is free from the classical paradoxes: \( A \land \neg A \models B \) and \( B \models A \lor \neg A \). I consider well-known extensions of FDE, which are formulated in terms of binary consequence systems: three-valued Kleene logic, three-valued Priest logic and classical logic. The first two of these can be semantically defined using semi-generalized state descriptions: for Kleene logic I use \( \top \)-generalized state descriptions (consistent but incomplete), for Priest logic I use \( \bot \)-generalized state descriptions (inconsistent but complete). The entailment relation for Kleene logic defined in terms of truth-and-non-falsity preservation from the premise to the conclusion. In turn Priest logic determined by entailment relation defined through the preservation of falsity-and-non-truth from the conclusion to the premise. The paper includes proofs of the corresponding completeness and soundness theorems. In the case of classical logic, we provide only a sketch of completeness and soundness with respect to the semantics of classical state descriptions (consistent and complete). This article is the first part of studies on E. K. Vojshvillo semantics for different extensions of FDE.

Keywords: Kleen logic, Priest logic, first-degree entailments, classical logic, generalized state descriptions

1. Introduction

The semantics of generalized state descriptions was proposed by E. K. Voishvillo in relation to the problem of interpreting the relevant entailment concept. The main methodological prerequisite for solving this problem may be considered

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to be Evgeny Voishvillo’s desire to express entailment as a connection between sentences on the inclusion of semantic information, which he understood as the true logical meaning of sentences. Therefore, this approach directly reveals the meaning of the term “relevant entailment”, unlike alternative approaches such as the WGS criterion, which Voishvillo considers to be: “intended to limit the classical entailment $A \models B$…” [18].

Semantic information of the sentence $A$ is contained in the definitions of logical constants in this sentence. However, as E. K. Voishvillo notes, additional suppositions of an ontological nature are included in truth conditions of classic logic formulas, namely on the consistency and completeness of those possible worlds to which these sentences relate. As a result of these suppositions we cannot grasp the pure logical meaning of sentences, since it is limited by these suppositions, which leads to the creation of sentences that do not make any sense. These sentences are formulas that form classic entailment paradoxes: $A \land \neg A$ and $A \lor \neg A$. Following on from this idea, Voishvillo suggests defining entailment through the subsumption of semantic information of the conclusion in the semantic information of the premise. If we denote a set in which an sentence $A$ is true as $M_A$ and a set of all possible outcomes as $M$, then the semantic information of an sentence $A$ is a pair $(M_A, M)$. In other words, semantic information of the sentence $A$ is a set which is true in relation to the set of all possible outcomes. Voishvillo views these “possible outcomes” as a state description. Assuming that $M$ can be infinite and therefore becomes universal for all sentences of the language, we can write down semantic information of an arbitrary sentence $A$ through $M_A$. Then the entailment relation is interpreted as “semantic information $B$ is a part of semantic information $A$”, symbolically represented:

$$A \models B \iff M_A \subseteq M_B$$

It is easy to see that under this interpretation of information, sentences of type $A \land \neg A$ and $A \lor \neg A$ either do not contain information at all (in the first case) or contain all possible information at once (the second case). These are the paradoxes that follow from those ontological suppositions which we shall consider further on. And they need to be rejected when building Voishvillo semantics to overcome classic entailment paradoxes.

In paper [14] E.D. Smirnova proposes a generalized approach to building semantics for intensional logics. In the same paper, the author considers an idea to extend Voishvillo’s semantics for a certain class of logic, namely: Hao Wang logic (denoted as $\text{HW}$); logic dual to Hao Wang logic (denoted as $\text{DHW}$); and the first-level fragment of Lukasiewicz logic. The same idea was expressed by Evgeny Voishvillo in paper [18].
The aim of this paper is to provide a thorough implementation of this idea. We present Voishvillo semantics based on the concept of the semi-generalized state description and prove their adequacy to the systems of the first-degree fragments of strong three-valued Kleene logic $K_3$ and Priest logic $P_3$, which are deductively equivalent to the systems of Hao Wang logic and dual to Hao Wang logic, respectively. Given that the semantics of the classic description of state is adequate for the first-degree fragment of the classic logic TV, we only give a brief description of the proof.

2. Semantics of semi-generalized state description for Kleene logic, Priest logic, and classic logic

All logics analyzed in this paper are based on the propositional language $L$, which we define based on the Backus–Naur form. Let $Prop$ denote a set of all propositional variables of the language $L$.

$$A := Prop \mid \neg A \mid (A \land A) \mid (A \lor A).$$

Let $Form$ denote a set of all formulas of the language $L$. We think of an evaluation function as the mapping $\upsilon: Prop \to \{t, f\}$. The set $\text{Literals} = Prop \cup \{\neg p: p \in Prop\}$ is a set of literals. A state description is thought of as such a (possibly infinite) set $\{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\} \subseteq \text{Literals}$, where each $\tilde{p}_i$ is a literal. A set of all state descriptions is denoted as $\text{States}$.

By classic state description we understand such a state description $\alpha$ that satisfies the following conditions:

I. $\forall p_i \in Prop \ (p_i \in \alpha \text{ or } \neg p_i \in \alpha)$;
II. $\forall p_i \in Prop \ \neg (p_i \in \alpha \text{ and } \neg p_i \in \alpha)$.

By generalized state description we understand such a state description $\alpha$ that does not meet conditions (I) and (II), i.e. it can be empty and contradictory.

By semi-generalized state description we understand $\top$-generalized state description and $\bot$-generalized state description. The former meets condition (II) but does not meet condition (I); the latter meets condition (I) but does not meet condition (II).

The idea of using the generalized state description to define the entailment relation between first-degree formulas$^1$ was initially mentioned in paper [17]. It is further developed in the well-known monograph [18]. In fact, the suggested

$^1$Formulas of type $A \rightarrow B$, where neither $A$ nor $B$ contain $\rightarrow$. In the considered logic, expressions $A \rightarrow B$ are equivalent to meta statements on derivability of $A \vdash B$, where $A$ and $B$ contain only connectives $\land, \lor, \neg$, respectively. This notation shall be used further in the paper. See for example: [19].
entailment relation is axiomatized by the equally well-known FDE system presented in the papers by A. Anderson and N. Belnap under the name 'tautological entailments' [4], [1], [2]. Adequate algebraic semantics was first proposed by M. Dunn in [6], where the four-valued De Morgan lattice is used as a model. This result was further developed by Josep Maria Font in the paper [9], where it was proved that the class of all De Morgan lattices was an algebraic counterpart of Belnap Dunn logic. By the late 1970s, Dunn’s ideas were developed in his own paper [7] and papers by N. Belnap [3], [5], where the famous four-valued semantics for 'tautological entailment' was suggested. This semantics is traditionally referred to as built according to the so-called American plan. An alternative approach was developed in papers by R. Routley and R. Meyer (see [12], [13]), where set-theoretic semantics for FDE is proposed. The Routley and Meyer approach is traditionally classified as semantics built according to the Australian plan. In this context, semantics by E.K. Voishvillo can be considered to be a kind of set-theoretic semantics according to the American plan (see [19]).

We apply semantics of generalized state description for FDE. Statements \( t \in \upsilon_\alpha(A) \) and \( f \in \upsilon_\alpha(A) \) mean that 'sentence \( A \) is true in state description \( \alpha \)' and 'sentence \( A \) is true in state description \( \alpha \)', respectively.

**Definition 1. (Truth conditions for FDE logic.)** We determine the function \( \upsilon: \text{States} \times \text{Prop} \to \{\emptyset, \{t\}, \{f\}, \{t, f\}\} \). We use \( \upsilon_\alpha \) as an abbreviation of \( \upsilon(\alpha, p_i) \), where \( \alpha \in \text{States}, p_i \in \text{Prop} \).

\[
\begin{align*}
t &\in \upsilon_\alpha(p_i) \iff p_i \in \alpha; \\
f &\in \upsilon_\alpha(p_i) \iff \sim p_i \in \alpha.
\end{align*}
\]

This function can be extended to the set of all formulas of our language as follows:

\[
\begin{align*}
t &\in \upsilon_\alpha(A \land B) \iff t \in \upsilon_\alpha(A) \text{ and } t \in \upsilon_\alpha(B); \\
t &\in \upsilon_\alpha(A \lor B) \iff t \in \upsilon_\alpha(A) \text{ or } t \in \upsilon_\alpha(B); \\
t &\in \upsilon_\alpha(\sim A) \iff f \in \upsilon_\alpha(A); \\
f &\in \upsilon_\alpha(A \land B) \iff f \in \upsilon_\alpha(A) \text{ or } f \in \upsilon_\alpha(B); \\
f &\in \upsilon_\alpha(A \lor B) \iff f \in \upsilon_\alpha(A) \text{ and } f \in \upsilon_\alpha(B); \\
f &\in \upsilon_\alpha(\sim A) \iff t \in \upsilon_\alpha(A).
\end{align*}
\]

**Definition 2.** \( \forall A, B \in \text{Form} \)

\[A \models_{\text{FDE}} B \iff \forall \alpha \in \text{States} (t \in \upsilon_\alpha(A) \Rightarrow t \in \upsilon_\alpha(B)). \]
Truth conditions following from definition 1 require a generalized state descriptions.

For proof that this semantics is adequate for the FDE system see [18].

In [8] M. Dunn describes a set of FDE extensions including first-degree fragments of Kleene strong logic $K_3$, Priest logic $P_3$, logic $RM$ and classic logic $TV$. These extensions are defined by attaching any combinations of well-known postulates to the list of FDE axiom schemes (for more details see [8]):

\[
A \land \neg A \vdash B \quad (\text{absurdity})
\]

\[
B \vdash A \lor \neg A \quad (\text{triviality})
\]

\[
A \land \neg A \vdash B \lor \neg B \quad (\text{safety})
\]

In this paper, adequate semantics are specified for $K_3$, $P_3$, and $TV$, which use a semi-generalized state descriptions (for $K_3$ and $P_3$) and classic state descriptions (for $TV$).

Truth conditions for logics $K_3$, $P_3$, and $TV$ are identical to those presented in definition 1. The only modification is the imposition of some limitations on the state description set, depending on which logic is used. If $\top$-generalized state description is used in definition 1, then we get truth conditions for $K_3$. If $\bot$-generalized state description is used, then we get truth conditions for $P_3$. If we limit ourselves to the classic state description, we get truth conditions for $TV$.

To make it easier to read the following proofs, we introduce the following notation. Let $|A|_\alpha = t$ be an abbreviation of $|A|_\alpha = t$ and an abbreviation of $(t \in v_\alpha(A)$ and $f \notin v_\alpha(A))$. Let $|A|_\alpha = f$ be an abbreviation of $(t \notin v_\alpha(A)$ and $f \in v_\alpha(A))$. Now we can put forward truth conditions for our logics as the following lemmas.

**Lemma 1.** Let $v_\alpha$ be the above defined evaluation function. Then for any evaluation $v_\alpha$, it holds true for any formula $A \in \text{Form}$:

\[
|\neg A|_\alpha = t \iff |A|_\alpha = f;
\]

\[
|\neg A|_\alpha = f \iff |A|_\alpha = t;
\]

\[
|A \land B|_\alpha = t \iff |A|_\alpha = t \text{ and } |B|_\alpha = t;
\]

\[
|A \land B|_\alpha = f \iff |A|_\alpha = f \text{ or } |B|_\alpha = f;
\]

\[
|A \lor B|_\alpha = t \iff |A|_\alpha = t \text{ or } |B|_\alpha = t;
\]

\[
|A \lor B|_\alpha = f \iff |A|_\alpha = f \text{ and } |B|_\alpha = f.
\]

We define the entailment relation.

**Definition 3.** \(\forall A, B \in \text{Form}\)

\(A \models_{K_3} B \iff \forall \alpha \in \text{States} \ (|A|_\alpha = t \implies |B|_\alpha = t)\).
Definition 4. \( \forall A, B \in \text{Form} \)

\[ A \models_{P_3} B \iff \forall \alpha \in \text{States} \ (|B|_\alpha = f \Rightarrow |A|_\alpha = f). \]

We call semantics based on definitions 1, 3 and 4, *Voishvillo semantics in the weak sense*.

3. Axiomatization

This chapter proposes an axiomatization of the considered logics in the form of binary consequence systems. This formalization method is widely used in logics with generalized truth values (see [15]), where it is considered as a universal method based on early works on first-degree relevant logic [1], [2]. However, in the papers mentioned above these calculations are referred to as 'Hilbert-style calculus' or 'Hilbert-style formalism'. The term 'Hilbert-style' requires some explanation in this case. As in the term’s standard meaning, we deal with systems consisting of axioms and inference rules. The only difference is that we will work with axioms not as with syntax analogies of tautologies, i.e. formulas which evaluated by designated value under any interpretation, with both syntax analogies of valid statements of entailment, i.e. such pairs of formulas where, if the designated value is attached to one of them (premise), the same assigned value is preserved for the second one (conclusion). As a result, inference rules should be viewed as deductibility sequences, i.e. relations of type \( A \vdash B \), with their premises and conclusion; in this case, rule validity is defined naturally — if premises are calculus theorems, then the conclusion is as well.

A set of formulas \( \beta \) is called theory in logic \( L \), if it meets conditions (1) and (2):

1. \( A \in \beta \) and \( B \in \beta \) \( \Rightarrow \) \( A \land B \in \beta \) (a closure with respect to conjunction);
2. If \( A \in \beta \) and \( A \vdash_L B \), than \( B \in \beta \) (a closure with respect to deductibility).

Theory \( \beta \) is called prime, it meets the condition:

3. \( A \lor B \in \beta \) \( \Rightarrow \) \( A \in \beta \) or \( B \in \beta \) (primeness).

Theory \( \beta \) is called consistent if it meets the condition:

4. \( A \in \beta \) \( \iff \) \( \neg A \notin \beta \) (consistency).

Logic system \( K_3 \) call a pair \((L, \vdash_{K_3})\), where \( L \) is the language used, and \( \vdash_{K_3} \) is the reflective relation which conforms to the following postulates and rules:

\[
a_1. A \land B \vdash_{K_3} A; \quad a_2. A \land B \vdash_{K_3} B; \quad a_3. A \vdash_{K_3} A \lor B;
\]
\[ a4. \, B \vdash_{\mathbf{K}_3} A \lor B; \quad a5. \, A \vdash_{\mathbf{K}_3} \neg \neg A; \quad a6. \, \neg \neg A \vdash_{\mathbf{K}_3} A; \]

\[ a7. \, A \land (B \lor C) \vdash_{\mathbf{K}_3} (A \land B) \lor (A \land C); \quad a8. \, \neg (A \land B) \vdash_{\mathbf{K}_3} \neg A \land \neg B; \]

\[ a9. \, \neg A \lor \neg B \vdash_{\mathbf{K}_3} \neg (A \land B); \quad a10. \, \neg (A \lor B) \vdash_{\mathbf{K}_3} \neg A \land \neg B; \]

\[ a11. \, \neg A \land \neg B \vdash_{\mathbf{K}_3} \neg (A \lor B); \quad a12. \, A \land \neg A \vdash_{\mathbf{K}_3} B; \]

\[ r1. \, A \vdash_{\mathbf{K}_3} B; \quad B \vdash_{\mathbf{K}_3} C \lor A \vdash_{\mathbf{K}_3} C; \]

\[ r2. \, A \vdash_{\mathbf{K}_3} B; \quad A \vdash_{\mathbf{K}_3} C \lor A \vdash_{\mathbf{K}_3} B \land C; \]

\[ r3. \, A \vdash_{\mathbf{K}_3} C; \quad B \vdash_{\mathbf{K}_3} C \lor A \lor B \vdash_{\mathbf{K}_3} C. \]

It is easy to prove the following proposition.

**Proposition 1.** \( \forall A, B \in \text{Form} \), \( (A \vdash_{\mathbf{K}_3} B) \iff (A \vdash_{\text{HW}} B) \).

**Proof.** We adopt definition \( \text{HW} \) from [14]. The theorem statement can be split into two statements:

(a). \( (A \vdash_{\text{HW}} B) \Rightarrow (A \vdash_{\mathbf{K}_3} B) \);

(b). \( (A \vdash_{\mathbf{K}_3} B) \Rightarrow (A \vdash_{\text{HW}} B) \).

\( \mathbf{K}_3 \) of two theorems, namely: \( A \land B \vdash_{\mathbf{K}_3} B \land A \) and \( A \lor B \vdash_{\mathbf{K}_3} B \lor A \). The first one is arrived at by sequential use of \( a2, a1, \) and rule \( r2 \), the second one by using \( a4, a3, \) and \( r3 \). For proof (b) proofs must be presented in system \( \text{HW} \) of two other theorems: \( B \vdash_{\text{HW}} A \lor B \) and \( A \land B \vdash_{\text{HW}} B \). The first one is arrived at by sequential use of \( A1, A2, \) and \( R3 \), the second one by using \( A4, A3, \) and \( R3 \).

\[ \square \]

**Definition 5.** For any \( p_i \in \text{Prop}, \) and for any prime theory \( \beta \) in logic \( L \) we define canonical valuation \( |.|_{\beta}^c \):

1. \( |p_i|_{\beta}^c = t \iff p_i \in \beta \) and \( \neg p_i \notin \beta \);
2. \( |p_i|_{\beta}^c = f \iff p_i \notin \beta \) and \( \neg p_i \in \beta \).

**Lemma 2.** Let \( |.|_{\beta}^c \) be a canonical valuation for logic \( \mathbf{K}_3 \). Then for any \( A \in \text{Form} \) the following holds:

1. \( |A|_{\beta}^c = t \iff A \in \beta \) and \( \neg A \notin \beta \);
2. \( |A|_{\beta}^c = f \iff A \notin \beta \) and \( \neg A \in \beta \).
PROOF. We prove it by induction using the number of logical connectives in the formula. The basic case when formula $A$ is a propositional variable is true due to Definition 5. Inductive assumption: let the statement of the lemma be true for formulas with the number of connectives less than $s$, where $s$ is the number of connectives in the formula under question.

(1). $\Rightarrow$. Let $|A|_\beta^c = t$. Then, by definition 1, we get $|A|_\beta^c = f$. From this, by inductive assumption, we arrive at $\sim A \in \beta$ and $A \notin \beta$. Using a6.

(2). $\Rightarrow$. Let $|\sim A|_\beta^c = f$. Then, by definition 1, we get $|A|_\beta^c = t$. From this, by inductive assumption we arrive at $A \notin \beta$. Using a5. $A \vdash_{K_3} \sim A$ and by definition of prime theory, we arrive at $\sim A \notin \beta$. By inductive assumption, we arrive at $|A|_\beta^c = f$. Following from definition 1, we arrive at $|\sim A|_\beta^c = t$.

(3). $\Rightarrow$. Let $|A \land B|_\beta^c = t$. Then by definition 1 we arrive at $|A|_\beta^c = t$ and $|B|_\beta^c = t$. From this, by inductive assumption we arrive at $A \in \beta$ and $\sim A \notin \beta$, and also $B \in \beta$ and $\sim B \notin \beta$. From this, by definition of the prime theory, we conclude that $A \land B \in \beta$. As $\sim A \notin \beta$ and $\sim B \notin \beta$ we arrive at $A \lor \sim B \notin \beta$. Using a8. $\sim (A \land B) \vdash_{K_3} \sim A \lor \sim B$ and by definition of the prime theory, we arrive at $\sim (A \land B) \notin \beta$.

(3). $\Leftarrow$. Let $A \land B \in \beta$ and $\sim (A \land B) \notin \beta$. As $A \land B \in \beta$, by definition of the prime theory, we arrive at $A \in \beta$ and $B \in \beta$. As $\sim (A \land B) \notin \beta$, using a9. $\sim A \lor \sim B \vdash_{K_3} \sim (A \land B)$ and by definition of the prime theory, we arrive at $\sim A \lor \sim B \notin \beta$. Therefore, $A \notin \beta$ and $\sim B \notin \beta$. Thus, by inductive assumption we arrive at $|A|_\beta^c = t$ and $|B|_\beta^c = t$, from where, by definition 1 $|A \land B|_\beta^c = t$ follows.

(4). $\Rightarrow$. Let $|A \land B|_\beta^c = f$. Then by definition 1 we arrive at $|A|_\beta^c = f$ or $|B|_\beta^c = f$. Case analysis.

Let $|A|_\beta^c = f$. Then by inductive assumption, we arrive at $\sim A \in \beta$ and $A \notin \beta$. Using a3. $\sim A \vdash_{K_3} \sim A \lor \sim B$, a9. $\sim A \lor \sim B \vdash_{K_3} \sim (A \land B)$, rule r1, and by definition of the prime theory, we arrive at $\sim (A \land B) \in \beta$. As $A \notin \beta$, using a1. $A \land B \vdash_{K_3} A$ and by definition of the prime theory, we arrive at $A \land B \notin \beta$.

Let $|B|_\beta^c = f$. Like in the previous case, we arrive at the desired $\sim (A \land B) \in \beta$ and $A \land B \notin \beta$.

(4). $\Leftarrow$. Let $\sim (A \land B) \in \beta$ and $A \land B \notin \beta$. Using a8. $\sim (A \land B) \vdash_{K_3} \sim A \lor \sim B$ and by definition of the prime theory, we arrive at $\sim A \lor \sim B \in \beta$, and therefore $\sim A \in \beta$ or $\sim B \in \beta$. Also, by definition of prime theory, we arrive at $A \notin \beta$ or $B \notin \beta$. Case analysis.
Let \( \sim A \in \beta \) and \( A \notin \beta \). Then, by inductive assumption, we arrive at \( |A|_{\beta}^c = f \). From here, by definition 1, we arrive at \( |A \land B|_{\beta}^c = f \).

Let \( \sim A \in \beta \) and \( A \notin \beta \). Using \( a2 \). \( A \land B \vdash_{K_3} B \) and by definition of the prime theory, we arrive at \( A \land B \notin \beta \), and therefore \( A \notin \beta \) and \( B \notin \beta \). As \( A \notin \beta \) and \( \sim A \in \beta \), by inductive assumption we arrive at \( A \land B \notin \beta \). From here, by definition 1, we arrive at \( |A \land B|_{\beta}^c = f \).

Let \( \sim B \in \beta \) and \( A \notin \beta \). Using \( a1 \). \( A \land B \vdash_{K_3} A \) and by definition of the prime theory we arrive at \( A \land B \notin \beta \), and therefore \( A \notin \beta \) and \( B \notin \beta \). As \( B \notin \beta \) and \( \sim B \in \beta \), by inductive assumption we arrive at \( A \land B \notin \beta \), and therefore \( A \notin \beta \) and \( B \notin \beta \). As \( B \notin \beta \) and \( \sim B \in \beta \), by inductive assumption we arrive at \( |A \land B|_{\beta}^c = f \). From here, by definition 1, we arrive at \( |A \land B|_{\beta}^c = f \).

Let \( \sim B \in \beta \) and \( B \notin \beta \). Like in the first case, here we have the desired statement \( |A \land B|_{\beta}^c = f \). Proof of the case when formula \( A \) looks like \( A \lor B \) is performed ambiguously in relation to cases when \( A = A \land B \)

\[ \square \]

**Lemma 3.** If \( A \not\vdash_{K_3} B \), then there exists a prime theory \( \alpha \), such that \( A \in \alpha \) and \( \sim A \notin \alpha \), and \( B \notin \alpha \).

**Proof.**

Our goal is to describe a procedure for designing a maximal theory. We shall use M. Dunn’s technique from [8].

We enumerate all formulas from the language in question \( A_0, A_1, A_2, \ldots \). We build a sequence of theories starting from \( \tau_0 = \{C \mid A \vdash_{K_3} C\} \). The following theories are built like this:

1. if \( \tau_n + A_{n+1} \vdash_{K_3} B \), then \( \tau_{n+1} = \tau_n \);
2. if \( \tau_n + A_{n+1} \not\vdash_{K_3} B \), then \( \tau_{n+1} = \tau_n \cup \{A_{n+1}\} \).

The resulting maximal theory \( \tau \) is arrived at by joining all \( \tau_n \)-s. \( \tau \) is closed under relation \( \vdash_{K_3} \) by its construction, since each \( \tau_n \)-s is closed under relation \( \vdash_{K_3} \).

The statement \( B \notin \tau \) also follows from the initial definition of \( \tau \), since otherwise we would conclude that \( A \vdash_{K_3} B \), and this contradicts the conditions of Lemma.

Let \( \tau \) be a inconsistent theory. Then there exists a formula \( C \), such that \( C \in \tau \) and \( \sim C \in \tau \). As \( \tau \) is closed under relation \( \vdash_{K_3} \), using \( a12 \) we arrive at \( C \land \sim C \vdash_{K_3} B \). As \( \tau \) is closed under relation \( \vdash_{K_3} \), we conclude that \( B \in \tau \). This is a contradiction, as we established previously that \( B \notin \tau \).

We only have to show that \( \tau \) has a property of primeness. We assume that \( D \lor E \in \tau \), but \( D \notin \tau \) and \( E \notin \tau \). By construction of \( \tau_n \)-th, we can conclude that \( \tau + D \vdash_{K_3} B \) and \( \tau + E \vdash_{K_3} B \). Consequently, there exists a formula
$C_1 \in \tau$, such that $C_1 \land D \vdash_{K_3} B$ and $C_1 \land E \vdash_{K_3} B$. By applying rule $r_3$, we conclude that $(C_1 \land D) \lor (C_1 \land E) \vdash_{K_3} B$, then using $a7$ and rule $r_1$, we arrive at $C_1 \land (D \lor E) \vdash_{K_3} B$. As theory $\tau$ is closed under relation $\vdash_{K_3}$, we arrive at $B \in \tau$, and this contradicts our initial assumption.

**Theorem 1.** (Completeness $K_3$). $\forall A, B \in \text{Form}$  
$(A \vDash_{K_3} B) \Rightarrow (A \vdash_{K_3} B)$.

**Proof.** We shall reason by contraposition. We assume that $A \not\vDash_{K_3} B$. Then, according to Lemma 3 we conclude that there exists prime theory $\alpha$, such that $A \in \alpha$ and $\sim A \notin \alpha$, and $B \notin \alpha$. According to Lemma 2 we arrive at $|A|_\alpha = t$ and $|B|_\alpha \neq t$. From here, using Definition 3, we arrive at $A \not\vDash_{K_3} B$.

**Theorem 2.** (Soundness $K_3$). $\forall A, B \in \text{Form}$  
$(A \vdash_{K_3} B) \Rightarrow (A \vDash_{K_3} B)$.

**Proof.** The proof of soundness is reduced to routine checking that all axioms $K_3$ are valid statements of entailment, and inference rules preserve this relation.

We consider axiom $a12. A \land \sim A \vdash_{K_3} B$.

1. $A \land \sim A \not\vDash_{K_3} B$ (assumption);
2. $|A \land \sim A|_\alpha = t$ and $|B|_\alpha \neq t$ (1, for some $\alpha$);
3. $|A|_\alpha = t$ and $|\sim A|_\alpha = t$ (2, Definition 1);
4. $|A|_\alpha = t$ and $|A|_\alpha = f$ (3, Definition 1);
5. $A \in \alpha$ and $\sim A \notin \alpha$ (4, Definition 1);
6. $A \notin \alpha$ and $\sim A \in \alpha$ (4, Definition 1).

Steps 5 and 6 contradict each other, therefore the initial assumption is wrong, and $A \land \sim A \not\vdash_{K_3} B$.

We consider the case of the rule $r1. A \vdash_{K_3} B; B \vdash_{K_3} C / A \vdash_{K_3} C$.

We assume that $A \vdash_{K_3} B; B \vdash_{K_3} C$ and $A \not\vdash_{K_3} C$. As $A \not\vDash_{K_3} C$, by definition of 3, we conclude that $|A|_\alpha = t$ and $|C|_\alpha \neq t$. Using $|A|_\alpha = t$, definition 3, $A \vdash_{K_3} B$ and $B \vdash_{K_3} C$, it is easy to get $|C|_\alpha = t$, that leads us to a contradiction.

**Corollary to Theorems 1 and 2:**

**Theorem 3.** (Adequacy $K_3$). $\forall A, B \in \text{Form}$  
$(A \vDash_{K_3} B) \iff (A \vdash_{K_3} B)$.

We refer to the pair $(\mathcal{L}, \vdash_{P_3})$ as logical system $P_3$, where $\mathcal{L}$ is the language we use, and $\vdash_{P_3}$ is a reflexive relation that satisfies the following postulates and rules:
\textbf{Proposition 2.} \forall A, B \in \text{Form} \\
(A \vdash_{P_3} B) \iff (A \vdash_{\text{DHW}} B).

\textbf{Proof.} The same as Proposition 1. \hfill \Box

\textbf{Lemma 4.} Let \(|.|_\beta^c\) be a canonical valuation for logic \(P_3\). Then for any \(A \in \text{Form}\) it holds true that:

1. \(|A|_\beta^c = t \iff A \in \beta \text{ and } \sim A \notin \beta\);
2. \(|A|_\beta^c = f \iff A \notin \beta \text{ and } \sim A \in \beta\).

\textbf{Proof.} \\
This lemma shall be proven in the same way as Lemma 2. \hfill \Box

\textbf{Lemma 5.} If \(A \not\models_{P_3} B\), then there exists a prime theory \(\alpha\), such that \(B \notin \alpha\) and \(\sim B \in \alpha\), and \(A \in \alpha\).

\textbf{Proof.} This lemma shall be proven in the same way as Lemma 3. \hfill \Box

\textbf{Theorem 4.} (Completeness \(P_3\)). \forall A, B \in \text{Form} \\
(A \vdash_{P_3} B) \Rightarrow (A \vdash_{P_3} B).

\textbf{Proof.} We shall reason by contraposition. We assume that \(A \not\models_{P_3} B\). Then, according to Lemma 5, we conclude that there exists prime theory \(\alpha\), such that \(B \notin \alpha\) and \(\sim B \in \alpha\), and \(A \in \alpha\). According to Lemma 4, we conclude that \(|B|_{\alpha} = f\) and \(|A|_{\alpha} \neq f\). From here, using Definition 4, we conclude that \(A \not\models_{P_3} B\). \hfill \Box
Theorem 5. (Soundness $P_3$). $\forall A, B \in Form$

$A \vdash_{P_3} B \Rightarrow (A \models_{P_3} B)$.

Proof. The same as theorem 2.

Theorems 4 and 5 have the following corollary:

Theorem 6. (Adequacy $P_3$). $\forall A, B \in Form$

$(A \models_{P_3} B) \iff (A \vdash_{P_3} B)$.

A similar result can be arrived at for the classical logic. It can be defined through addition, for example, of axiom $B \vdash A \lor \neg A$ to the system $K_3$. To prove adequacy, it is sufficient to accept ontological postulates (I) and (II) for state description and to specify entailment in line with one of the definitions 3 and 4, based on the same truth conditions as in definition 1. The canonical estimator is defined in the same way as for $K_3$ and $P_3$. An analogue of Lindenbaum’s lemma looks as follows.

Lemma 6. We assume that $A \not\models_{TV} B$. Then there exists a prime theory $\theta$, such that $A \in \theta$ and $\neg A \notin \theta$, and $B \notin \theta$ or $\neg B \in \theta$.

It is easy to prove the following theorems.

Theorem 7. (Completeness $TV$). $\forall A, B \in Form$

$(A \models_{TV} B) \Rightarrow (A \vdash_{TV} B)$.

Theorem 8. Soundness $TV$). $\forall A, B \in Form$

$(A \vdash_{TV} B) \Rightarrow (A \models_{TV} B)$.

Theorem 9. (Adequacy $TV$). $\forall A, B \in Form$

$(A \models_{TV} B) \iff (A \vdash_{TV} B)$.

4. Conclusion

So, adequate semantics of semi-generalized state for logics $K_3$ and $P_3$ and, consequently, for logics $HW$ and $DHW$ are presented above. It is worth noting that if we specify entailment with the aid of definitions 3, using generalized state description, we will get semantics for $ETL$ logic. Semantically, this logic could be defined if we consider $FDE$ with the sole designated value — $T$ (‘told Truth’, see [3]). This logic is presented for the first time in [11]. If we specify entailment with definition 4, using the generalized state description, we will get semantics for $NFL$ logic. In [16], this logic is considered as $FDE$ logic with three designated values: $T$ (‘told Truth’), $B$ (‘told Truth and False’), $N$ (‘neither Truth, nor False’). Proof of adequacy of generalized state descriptions semantics for logics $ETL$ and $NFL$ will be presented in the follow-up paper.
References


В настоящей статье исследуются семантики полуобобщенных описаний состояний, которые являются разновидностью информационной семантики Е.К. Войшвилло, предложенной им для первоуровневой релевантной логики (FDE) в начале восьмидесятых годов. Ключевой особенностью войшвилловского подхода является рассмотрение описаний состояний, на которые не налагаются классические условия о непротиворечивости и полноте, что позволяет определить релевантное отношение следования. Под релевантным отношением следования понимается такое, для которого не проходят классические парадоксы: $A \land \neg A \models B$ и $B \models A \lor \neg A$. Нами рассматриваются известные расширения логики FDE, сформулированные в терминах систем бинарных следований: трехзначная логика Клини, логика Приста и классическая логика. Первые две из них могут быть семантически определены при помощи полуобобщенных описаний состояний: для логики Клини вводится понятие $\top$-обобщенных описаний состояний (непротиворечивых, но неполных), для логики Приста используется понятие $\bot$-обобщенных описаний состояний (противоречивых, но полных). Отношение следования, порождающее логику Клини, определяется через сохранность истинности и не-ложности от посылки к заключению. В свою очередь, логика Приста определяется отношением следования через сохранность ложности и не-истинности от заключения к посылке. В статье предлагаются доказательства адекватности данных семантик указанным системам. В случае с классической логикой мы формулируем лишь набросок доказательства полноты и непротиворечивости относительно семантики с классическими описаниями состояний (непротиворечивыми и полными). Настоящая статья является первой частью исследования, посвященного семантикам Е.К. Войшвилло для расширений логики FDE.

Keywords: логика Клини, логика Приста, классическая логика, первоуровневая релевантная логика, описания состояний, информационная семантика

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