Logical Matrices and Goldbach Problem

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The paper deals with equivalent formulations of Goldbach conjecture in terms of sets of tautologies in sequences of logical matrices and singular logical matrices. The significant role in this consideration belongs to concepts of tautology in a logical matrix, sums and products of logical matrices from the sequence $K_{n+1}$ of Karpenko matrices. Thus the paper proposes an answer to A.S. Karpenko’s question regarding possible relations between sequences of logical matrices similar to $K_{n+1}$ and an open problem, known as binary Goldbach conjecture: every even natural number $n \geq 4$ may be represented as a sum of two prime numbers. We prove that all finite-valued matrices in the sequence $M$ have tautologies iff the binary version of Goldbach conjecture ($G_2$) is true. Using the properties of matrix product operation, it is proven that the infinite-valued matrix $M \otimes$ has tautologies iff $G_2$ is true. It is shown that $G_2$ is equivalent to the validity of statement of identity between the set of tautologies of the matrix $M \otimes$, which constitutes the logical theory defined by this matrix, and a logical theory defined in terms of sets of tautologies in Łukasiewicz’s finite-valued logics $L_n$. These results were restated in terms of sequences of matrices and their products from a large class of logical matrices. Thus it was established that constructions utilizing the sequence $K_{n+1}$ can be seen as a particular case of constructions in such classes. The Goldbach conjecture then takes on the logical aspects, since the question of its validity or invalidity now narrows down to the question of nonemptiness of a certain logical theory.

Keywords: many-valued logics, logical matrices, tautologies, Goldbach conjecture

1. Introduction

Several equivalent statements of Goldbach’s conjecture are presented in this paper in terms of tautology sets generated by sequences of logical matrices and singular logical matrices.

Goldbach’s conjecture is named after Christian Goldbach who posed it in his letter to Leonhard Euler in 1742 (see, for example [13]). There are two known versions of this conjecture — binary and ternary. The binary statement of Goldbach’s conjecture is: any even natural number $n \geq 4$ can be written as the sum of two prime numbers. Hereinafter this statement will be denoted as $G_2$. The following statement is the ternary version of Goldbach’s conjecture

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(abbreviated: $G_3$): any odd integer $m \geq 7$ can be written as the sum of three prime numbers.

Since any odd natural number $m \geq 7$ can be written as $2k + 3$, it is obvious that the validity of $G_2$ implies the validity of $G_3$. However, the reverse implication could be false. This paper considers only the binary version; however, all submitted results can be easily extended to the ternary statement as well.

The Goldbach ternary conjecture was proved by Harald Helfgott in 2013 ([3] and [4]). Book [12, p. 39] provides the following information with regard to the efforts which have been exerted for more than 250 years in order to prove Goldbach’s binary hypothesis: “Recent years demonstrate huge progress in solving this problem, however, it has still not been solved.”

The question concerning possible connection between logical matrices and Goldbach’s conjecture was raised by A.S. Karpenko in [5] and [9]. Various many-valued logics and corresponding logical matrices are thoroughly studied in the monographs [6] and [8].

2. Logical matrices, Tautologies and Functional Equivalence

Further developments will require some definitions and results. In particular, the logical matrix, propositional tautology and functional equivalency concepts play the key role in the issues in focus.

**Definition 1.** A logical matrix is an ordered triple

$$\mathfrak{M} = \langle A, F, D \rangle,$$

where:

- $A$ is a set called the matrix carrier;
- $F = \{ f_{k_1}^1, f_{k_2}^2, \ldots, f_{k_m}^m \}$ is a set of completely defined functions

$$f_{k_i}^i : A^{k_i} \to A$$

for all $1 \leq i \leq m$;

- $D \subset A$ is a nonempty set of designated values (or elements).

**Definition 2.**

- A propositional tautology (or simply “tautology”) in the matrix $\mathfrak{M}$ is a formula $\varphi$ of the standard propositional language$^1$, such that the function

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$^1$We call a language standard, if its well-formed expressions are finite and constructed from a countably infinite set of different propositional variable using function symbols from $F$. 

represented by such a formula takes on a value from \( D \) with any set of variable values from \( A \) included in the function.

- The set of all tautologies of the matrix \( \mathcal{M} \) will be denoted as \( E(\mathcal{M}) \).

**Definition 3.** The matrices \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) with the common carrier set \( A \) are called functionally equivalent (notation: \( \mathcal{M}_1 \equiv \mathcal{M}_2 \)), if the following two conditions are fulfilled:

- for any function \( f_{i}^{k_i} (1 \leq i \leq m_1) \) from \( \mathcal{M}_1 \) there is formula \( \varphi(x_1, \ldots, x_{k_i}) \) in the language of the matrix \( \mathcal{M}_2 \) so that the function which corresponds to it is equal to \( f_{i}^{k_i} \);
- for any function \( g_{i}^{q_i} (1 \leq i \leq m_2) \) from \( \mathcal{M}_2 \) there is formula \( \psi(x_1, \ldots, x_{q_i}) \) in the language of the matrix \( \mathcal{M}_1 \) so that the function which corresponds to it is equal to \( g_{i}^{q_i} \).

It should be pointed out that functionally equivalent matrices are different variants for defining the same set of functions since, according to the definition, the sets of the functions that can be expressed in them coincide. The functional equivalency of two matrices with the same sets of designated elements also means that there is a possibility to identify the tautology sets of these matrices with respect to a specifically defined function for translation of expressions from the language of the first matrix into the language of the second one, and vice versa. More detailed information on this matter is available in the works [14], [15], [16] and [1].

In particular, if \( \mathcal{M}_1 \equiv \mathcal{M}_2 \), \( D_1 = D_2 \), then:

\[ E(\mathcal{M}_1) \neq \emptyset \iff E(\mathcal{M}_2) \neq \emptyset. \]

### 3. Logical Matrix and Karpenko Theorem Sequences

Lukasiewicz finite-valued logic is an example of the logical matrix. In this case, for any natural \( n \) a separate matrix \( L_{n+1} \) is defined; it is called Lukasiewicz \( n + 1 \)-valued logic. Lukasiewicz logics were introduced by Jan Lukasiewicz in 1918 (see [10] and [11]).

**Definition 4.** Lukasiewicz finite-valued logic is a logical matrix

\[ L_{n+1} = \langle V_{n+1}, \neg, \rightarrow, \{n\} \rangle, \]

where:

- \( V_{n+1} = \{0, 1, \ldots, n\} \);
• $\sim x = n - x$;

• $x \rightarrow y = \begin{cases} n, & \text{if } x \leq y; \\ n - x + y, & \text{if } x > y. \end{cases}$

In [7] A.S. Karpenko developed a sequence of matrices $K_{n+1}$ ($n \geq 3$) with extraordinary properties. Let us provide their definition.

**Definition 5.** A Karpenko matrix is a logical matrix $K_{n+1} = \langle V_{n+1}, \sim, \rightarrow^K, \{n\} \rangle$, where:

• $V_{n+1} = \{0, 1, \ldots, n\}$;

• $\sim x = n - x$;

• $x \rightarrow^K y = \begin{cases} y, & \text{if } 0 < x < y < n \text{ and } (x, y) \neq 1; \\ y, & \text{if } 0 < x = y < n; \\ x \rightarrow y, & \text{if otherwise}. \end{cases}$

In this definition the notation $(x, y) \neq 1$ means that the numbers $x$ and $y$ are not coprime numbers, i.e. they have common divisors other than one; $x \rightarrow$ is the value of the corresponding function $L_{n+1}$.

There are several results regarding the connection between Lukasiewicz finite-valued logics and prime numbers ($\Pi$ will stand for a set of all prime numbers). In particular, V.K. Finn proved [2] that the logic $L_{n+1}$ is pre-complete if and only if $n \in \Pi$.

Matrices $K_{n+1}$ are linked to prime numbers and Lukasiewicz logics by the two theorems proved by A.S. Karpenko (see [5]), which are presented below without proof.

**Theorem 1.** For any natural number $n \geq 3$ the following statement is true

$$n \in \Pi \Leftrightarrow E(K_{n+1}) \neq \emptyset.$$  

**Theorem 2.** For any natural number $n \geq 3$ it is true that $n \in \Pi$, if and only if the matrices $K_{n+1}$ and $L_{n+1}$ are functionally equivalent.

It should be noted that theorem 1 can be derived from theorem 2 in conjunction with the fact that $E(L_{n+1}) \neq \emptyset$ for all $n \geq 2$ and the one-element set $\{n\}$ is a set of designated values both in $K_{n+1}$ and $L_{n+1}$.
4. Products and Sums of Logical Matrices

Product and sum operations for logical matrices defined below play a key role in subsequent development.

**Definition 6.** The product of logical matrices
\[ M_1 = \langle A_1, f_1^{k_1}, \ldots, f_m^{k_m}, D_1 \rangle \]
and
\[ M_2 = \langle A_2, g_1^{k_1}, \ldots, g_m^{k_m}, D_2 \rangle \]
is the matrix
\[ M_1 \otimes M_2 = \langle A_1 \times A_2, h_1^{k_1}, \ldots, h_m^{k_m}, D_1 \times D_2 \rangle, \]
where the following is true for all \( i \) (\( 1 \leq i \leq m \)):
\[ h_i^{k_i}(<a_1^1, a_1^2>, <a_1^2, a_2^2>, \ldots, <a_i^k, a_i^k>) = <f_i^{k_i}(a_1^1, \ldots, a_i^k), g_i^{k_i}(a_1^1, \ldots, a_i^k)> \].

**Definition 7.** The sum of the logical matrices \( M_1 \) and \( M_2 \) is the matrix
\[ M_1 \oplus M_2 = \langle A_1 \times A_2, h_1^{k_1}, \ldots, h_m^{k_m}, D' \rangle, \]
where \( h_i^{k_i} \) for all \( i \) (\( 1 \leq i \leq m \)) is defined the same way as in the product operation, while
\[ D' = \{<a_1, a_2> : a_1 \in D_1 \lor a_2 \in D_2\}. \]

**Remark 1.**
- It follows from the definition of the matrix product that for every family of logical matrices \( \{M_i : i \in \alpha\} \) (\( \alpha \) is a finite or infinite ordinal) it holds that
\[ E(\otimes_{i=1}^{\alpha} M_i) \neq \emptyset, \]
if and only if it is the case that \( E(M_i) \neq \emptyset \) for each \( i \in \alpha \).
- Similarly, it follows from the definition of matrix sum that for every family of logical matrices \( \{M_i : i \in \alpha\} \) (\( \alpha \) is a finite or infinite ordinal) it holds that
\[ E(\oplus_{i=1}^{\alpha} M_i) \neq \emptyset, \]
if and only if \( E(M_i) \neq \emptyset \) for some \( i \in \alpha \).
5. Statements regarding Logical Matrices Sequences, Sums and Products

Application of sum and product operations to finite and infinite sets (or sequences) of logical matrices makes it possible to formulate the principal results of this paper. The characteristics of tautology sets in the resulting matrices are determined by the properties of these operations presented in note 1 as well as in theorems 1 and 2.

From now and to the end of the section we shall assume that $K_3$ is $L_3$.

Accounting for this assumption, let us consider the matrix sequence

$$M = M_2, M_3, M_4, \ldots,$$

where $M_2$ is $L_3$, and with $3 \leq j < \omega$:

$$M_j = \oplus_{i=2}^{j}(K_{i+1} \otimes K_{2j-i+1}).$$

**Remark 2.** With reference to Goldbach’s conjecture, every matrix $M_j$ ($3 \leq j < \omega$) of the set $M$ represents an even number $2j$. For example:

$$M_3 = (K_3 \otimes K_5) \oplus (K_4 \otimes K_4)$$

and

$$M_4 = (K_3 \otimes K_7) \oplus (K_4 \otimes K_6) \oplus (K_5 \otimes K_5)$$

etc.

Here every member of the sum corresponds to one of the representations of the number $2j$ as a sum of two smaller numbers $i$ and $2j - i$, while the whole sum contains complete enumeration of such possible pairs corresponding to the pairs of matrices $K_{i+1}$ and $K_{2j-i+1}$ respectively.

The condition above can serve as a basis when proving the following theorem.

**Theorem 3.** The sets $E(M_j)$ of the sequence $M$ are not empty for all $j$ ($2 \leq j < \omega$) if and only if the statement $G_2$ is true.

**Proof.**

($\Rightarrow$)

Let us assume that for any $j$ ($2 \leq j < \omega$) the set $E(M_j)$ is non-empty.

As $E(M_j) \neq \emptyset$ then, according to the definition of a logical matrix, there exists such $i$ ($2 \leq i \leq j$) that the summand $K_{i+1} \otimes K_{2j-i+1}$ has tautologies. Therefore, by the definition of a matrix product, it follows that both $E(K_{i+1})$ and $E(K_{2j-i+1})$ are non-empty. Based on theorem 1, this means that $i$ as well as $2j - i$ are prime numbers. It is obvious that $2j = i + 2j - i$, i.e. statement of Goldbach’s hypothesis, is true for $2j$. 
As the matrices in the set $M$ represent all natural numbers of the $2j$ type at $j \geq 2$, then the statement $G_2$ is true.

The proof ($\Rightarrow$) is complete.

(\Leftarrow)

Assume that the statement $G_2$ is true and there is the condition $j$ \(2 \leq j < \omega\) that $E(\mathcal{M}_j) = \emptyset$.

As the set $E(\mathcal{M}_j)$ is empty, then, according to the definition of a logical matrix, for any $i$ \(2 \leq i \leq j\), the $K_{i+1} \otimes K_{2j-i+1}$ summands have no tautologies. By definition of the matrix product, this means that either $E(K_{i+1}) = \emptyset$ or $E(K_{2j-i+1}) = \emptyset$. But in this case, based on theorem 1, at least one of the numbers $i$ and $j-i$ is not a prime number. Therefore, the number $2j$ is a counter-example for $G_2$; however, this contradicts with the assumption.

The proof ($\Leftarrow$) is complete.

Now we shall consider the matrix

$$M \otimes = \mathcal{M}_2 \otimes \mathcal{M}_3 \otimes \ldots = \otimes_{j=2}^{\omega} \mathcal{M}_j.$$ 

The following statement is true for this matrix.

**Theorem 4.** The set $E(M \otimes)$ is non-empty if and only if the statement $G_2$ is true.

**Proof.** The proof is derived from the previous theorem and the logical matrix product definition. \(\square\)

**Remark 3.** In terms of tautology sets generated by Karpenko matrices, the set $E(M \otimes)$ can be represented as follows:

$$E(L_3) \cap \bigcap_{j=3}^{\omega} \bigcup_{i=2}^{j} (E(K_{i+1}) \cap E(K_{2j-i+1})).$$

In view of the above, Goldbach’s statement is true if and only if the following equation is holds:

$$E(M \otimes) = E(L_3) \cap \bigcap_{j=3}^{\omega} \bigcup_{r=1}^{k_j} (E(L_{i_r+1}) \cap E(L_{2j-i_r+1})).$$

where

$$k_j = |\{<i,2j-i> : 2 \leq i \leq j \land i \in \Pi \land 2j-i \in \Pi\}|,$$

while the infinite intersection includes only the cases where $k_j \neq 0$. 

6. Statements on Logical Matrix Classes

Theorems 3 and 4 are of a rather particular nature and can be generalized to statements regarding quite a wide range of logical matrix classes that comply with a highly “compact” set of conditions.

In order to do that, we shall analyze the logical matrix class $\mathcal{A}$ which satisfies two conditions:

- $\mathcal{A}$ is closed with respect to finite and infinite sums and products of its elements;
- $\mathcal{A}$ contains at least two matrices $\mathfrak{N}_1$ and $\mathfrak{N}_2$ such that $E(\mathfrak{N}_1) = \emptyset$ and $E(\mathfrak{N}_2) \neq \emptyset$.

Let $[\mathfrak{N}_1]$ be the class of all matrices from $\mathcal{A}$ the tautology sets of which are empty. Let us denote the matrix class with non-empty tautology sets as $[\mathfrak{N}_2]$.

Now let

$$M = \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4, \ldots$$

be the set of matrices $\mathfrak{M}_j \in \mathcal{A}$ which fulfills the condition that for any natural number $j \geq 2$ the following is true:

- $\mathfrak{M}_j \in [\mathfrak{N}_2]$, if $k_j \neq 0$;
- $\mathfrak{M}_j \in [\mathfrak{N}_1]$ otherwise.

Like in the previous section $M \otimes$ is $\otimes_{j=2}^{\omega} \mathfrak{M}_j$. In this case the following statements are true.

**Theorem 5.** Tautology sets of the matrices $\mathfrak{M}_j \in \mathcal{A}$ that form the sequence $M$ are non-empty for any $2 \leq j < \omega$ if and only if the statement $G_2$ is true.

**Proof.** The validity of this theorem is a direct corollary of the definitions of the class $\mathcal{A}$, number $k_j$ and sequence $M$. \qed

**Theorem 6.** The tautology set of the logical matrix $M \otimes$ is non-empty if and only if the statement $G_2$ is true.

**Proof.** The validity of this theorem is a direct corollary of the definitions of the class $\mathcal{A}$, number $k_j$ and $M$, and the properties of matrix product operation. \qed

Let us analyze the sequence

$$K = \mathfrak{K}_3, \mathfrak{K}_4, \mathfrak{K}_5, \ldots$$

of logical matrices $\mathcal{A}$, such that the following two conditions are fulfilled:
\[ K_{r+1} \in [\mathcal{N}_2], \text{ if } r \in \Pi; \]

\[ K_{r+1} \in [\mathcal{N}_1] \text{ otherwise.} \]

It should be noted that the matrix sequence \( K_{n+1} \) (Karpenko matrix) is a special case of the sequences defined now.

If now \( M = M_2, M_3, M_4, \ldots \) and

\[ M_j = \bigoplus_{i=2}^{j} (K_{i+1} \otimes K_{2j-i+1}), \]

where \( j \geq 2 \), then:

\[ E(M \otimes) = \bigcap_{j=2}^{\omega} \left( \bigcup_{i=2}^{j} (E(K_{i+1}) \cap E(K_{2j-i+1})) \right). \]

Therefore, Goldbach's hypothesis has an equivalent formulation as the statement regarding non-emptiness of the tautology set generated by a matrix from a rather broadly defined class \( \mathcal{A} \) of logical matrices.

Goldbach's conjecture itself turns to a "logical domain" as its validity is now analyzed in conjunction with a logical theory with a specific set of properties. This statement is valid for both \( G_2 \) and \( G_3 \). By modifying the definitions of the matrix sequence \( M \), a logical theory can be developed to characterize \( G_3 \). As \( G_3 \) is true, the matrix \( M \otimes \) which corresponds to such sequence has the non-empty tautology set.

Notice that the class \( \mathcal{A} \) is a proper class which means that it has an enormous "size". So, for example, the set of \( \mathcal{A} \) can not be characterized by any cardinal \( \kappa \).

Assume that \( |\mathcal{A}| = \kappa \). In this case, according to note 1, we shall consider the product of all elements \( M_i \in \mathcal{A} \), where \( i < \alpha \), and \( \alpha \) is an ordinal isomorphic to \( \kappa \).

Then, based on the conditions applied to \( \mathcal{A} \), \( \otimes_{i=1}^{\alpha} M_i \in \mathcal{A} \). However, this contradicts the assumption of a one-to-one correspondence between \( \mathcal{A} \) and \( \kappa \), as this matrix does not coincide with any matrix from \( \mathcal{A} \) corresponding to \( \kappa \).

This points to a sufficiently non-trivial combinatorial nature of arithmetic claims similar to \( G_2 \).

References


В статье рассматриваются эквивалентные формулировки бинарной проблемы Гольдбаха в терминах множеств тавтологий последовательностей логических матриц и отдельных логических матриц. При этом существенную роль играют понятия тавтологий логических матриц, а также произведений и сумм логических матриц из последовательности $K_{n+1}$ (матриц Карпенко). Таким образом, в статье дается вариант ответа на поставленный А.С. Карпенко вопрос о возможности наличия связи между подобными $K_{n+1}$ последовательностями матриц и отдельными логическими матрицами и известной как бинарное утверждение Гольдбаха открытой проблемой: всякое четное натуральное число $n \geq 4$ может быть представлено в виде суммы двух простых чисел ($G_2$). Доказано утверждение о том, что всякая конечнозначная матрица в построенной последовательности $M$ имеет тавтологии, если и только если $G_2$ является истинным. С использованием свойств операции произведения матриц доказано, что бесконечнозначная матрица $M \otimes$ имеет тавтологии, если и только если $G_2$ истинно. Показано, что $G_2$ эквивалентна верности утверждения о равенстве множества тавтологий матрицы $M \otimes$, образующего заданную этой матрицей логическую теорию, и логической теории, определенной в терминах множеств тавтологий конечнозначных логик Лукасевича $L_n$. Данные результаты распространены на последовательности матриц и произведения матриц из таких последовательностей, входящие в довольно широкую совокупность классов матриц. За счет этого установлено, что построения с использованием последовательности $K_{n+1}$ могут рассматриваться в качестве частного случая построений в данных классах. Проблема Гольдбаха таким образом приобретает логические аспекты, так как вопрос о ее истинности или ложности теперь сводится к вопросу о непустоте определенной логической теории.

Ключевые слова: многозначная логика, логические матрицы, тавтологии, проблема Гольдбаха
Логические матрицы и проблема Гольдбаха

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