On properties of a class of four-valued paranormal logics

Tomova Natalya Evgen’evna
Institute of Philosophy of Russian Academy of Sciences.
12/1 Goncharnaya Str., Moscow, 109240, Russian Federation.
E-mail: natalya-tomova@yandex.ru

The paper is devoted to the results obtained during the investigation of a class of four-valued literal paranormal logics, i.e. logics, which are both paraconsistent and paracomplete at the level of literals; that is, formulas that are propositional variables or their iterated negations. Paraconsistent logic allows the possibility of operating with inconsistent information, paracomplete logic allows us to build reasoning in conditions of incomplete information. Paranormal systems allow to work with both types of uncertainty, information that is both inconsistent and incomplete. In [10] the class of four-valued literal paralogics obtained by combining isomorphs of classical logic, which are contained in Bochvar’s four-valued logic \( \mathbb{B}_4 \), is considered. It was found that together with the isomorphs themselves, logical matrices that correspond to these logics form a ten-element upper semilattice with respect to the functional embedding of one matrix into another. In this paper we investigate the class of matrices that make up the supremum of the said semilattice. The matrices of this class have interesting functional properties, namely, they correspond to the class of all external four-valued functions. The paper also provides an algorithm for constructing a perfect disjunctive \( J \)-normal form of a four-valued external function. As it turned out, there are well-known logics in the literature that are functionally equivalent to the logics of the class in question. For example, one of them is the logic \( \mathbb{V} \) [17], which is a formalization of intuitions of N.A. Vasilyev’s imaginary logic. We have considered the question of the correlation of all these systems both in the class of tautologies and in the class of valid consequence relations. As a result, it is proved that all systems are equivalent with respect to classes of tautologies, but they differ in the properties of their consequences relations.

Keywords: four-valued paranormal logics, functional properties, external functions, tautologies, consequence relation

1. Introduction

The need to work correctly in the context of inconsistent and incomplete information gave rise to the development of paralogic systems. Paralogics include paraconsistent, paracomplete and paranormal systems. Before providing definitions below, it should be pointed out that paraconsistent logics account for the possibility of operating with inconsistent information; paracomplete
logics allow us to construct reasoning when there is incomplete information. Paranormal systems allow to work with both types of uncertainty, information that is both inconsistent and incomplete. This paper deals with the study of the properties of a class of four-valued paranormal logics. This class shall be analyzed from the point of view of their functional properties, classes of tautologies and consequence relations.

2. Definitions

There are several approaches to the representation and analysis of logical systems. In this paper, logical systems shall be represented by means of logical matrices. Let us introduce some basic definitions.

**Definition 1.** Let $\text{Var} = \{p, q, r, \ldots\}$ be a countable set of propositional variables, and let $\text{Con} = \{F_1, \ldots, F_n\}$ be a finite set of propositional connectives, where each connective $F_i$ is associated with the natural number $a(F_i)$, which denotes the number of its arguments. For at least one $i \in \{1, \ldots, n\}$ it holds that $a(F_i) \neq 0$. The set $\text{For}$ is defined inductively:

1. $\text{Var} \subseteq \text{For}$,
2. For each such $F_i \in \text{Con}$, that $a(F_i) = k$, $F_i(A_1, \ldots, A_k) \in \text{For}$, if $A_1, \ldots, A_k \in \text{For}$,
3. Nothing else belongs to $\text{For}$.

The algebra of formulas $\mathcal{L} = \langle \text{For}, F_1, \ldots, F_m \rangle$ shall be called a propositional language.

Let $\mathcal{A} = \langle V, f_1, \ldots, f_m \rangle$ be an algebra of the same type as the propositional language $\mathcal{L}$, where $V$ is a set of truth values and $f_i$ is a function on $V$ of the same arity as $F_i$.

**Definition 2.** An ordered pair $\mathcal{M} = \langle \mathcal{A}, D \rangle$, where $\mathcal{A}$ is an algebra of the same type as the propositional language $\mathcal{L}$ and $D \subseteq V$ is a nonempty proper subset of $V$, is called the logical matrix for $\mathcal{L}$. Elements of $D$ are called designated values of $\mathcal{M}$.

**Definition 3.** A valuation $v$ of the formula $A$ in the matrix $\mathcal{M}$ for the language $\mathcal{L}$ is a homomorphism from $\mathcal{L}$ into $\mathcal{A} = \langle V, f_1, \ldots, f_m \rangle$, such that

1. if $p$ is a propositional variable, then $v(p) \in V$;
2. if $A_1, A_2, \ldots, A_n$ are formulas and $F^n$ is an $n$-ary connective of language $\mathcal{L}$, then $v(F^n(A_1, A_2, \ldots, A_n)) = f^n(v(A_1), v(A_2), \ldots, v(A_n))$, where $f^n$ is a function on $V$ corresponding to $F^n$.

**Definition 4.** Some formula $A$ is a tautology in $\mathcal{M}$ (abbreviated to $\models_\mathcal{M} A$), iff for every valuation $v$ in $\mathcal{M}$ it is true that $v(A) \in D$. 
DEFINITION 5. The theory generated by $\mathcal{M}$ is the set of all tautologies in $\mathcal{M}$. It is denoted by $E(\mathcal{M})$.

DEFINITION 6. The formula $B$ logically follows from the set of formulas $\Gamma = \{A_1, A_2, \ldots, A_n\}$ in $\mathcal{M}$ (abbreviated to $\Gamma \models_{\mathcal{M}} B$), iff there is no such valuation $v$ in $\mathcal{M}$, such that $v(A_i) \in D$ for each $A_i \in \Gamma$ and $v(B) \notin D$.

DEFINITION 7. The consequence relation generated by $\mathcal{M}$ is the set $Cn(\mathcal{M})$ of ordered pairs $\langle \Gamma, B \rangle$, such that for every valuation $v$ in $\mathcal{M}$ if $v(\Gamma) \subseteq D$, then $v(B) \in D$.

Next, we shall present some definitions concerning paralogics. There are various formal and substantive criteria that characterize paraconsistency, paracompleteness and paranormality.

It is necessary to take into account that matrix logics can be analyzed using different approaches [6, p. 30–32].

If we regard a logic as a theory, that is, as a class of tautologies, then S. Jaskowski’s “implicative-negative” criterion of paraconsistency [9] is convenient: In the system of paraconsistent logic, the Duns Scotus law $A \supset (\neg A \supset B)$ fails.

In the paracomplete logic system, the Clavius law $\neg A \supset A \supset A$ [4] fails.

Other authors, for example [15], [17, p. 210], define paraconsistency and paracompleteness as follows. A theory $T$ of logic $L$ is called trivial if all formulas of $L$ are theorems of $T$, otherwise the theory is nontrivial. An inconsistent theory of logic $L$ is a theory $T$ of logic $L$, such that for some formula $A$ the following is true: $A$ and $\neg A$ are theorems $T$. A paraconsistent theory of logic $L$ is an inconsistent theory $T$ of logic $L$, such that $T$ is not a trivial theory. By a complete theory of logic $L$ we mean a theory $T$ of logic $L$, such that for every formula $A$ the following is true: $A$ or $\neg A$ is a theorem in $T$. A paracomplete theory of logic $L$ is a theory $T$ of logic $L$, such that $T$ is not a complete theory of logic $L$ and every complete theory of logic $L$, which contains $T$, is a trivial theory.

If a logic is seen as a class of inferences, then logic is paraconsistent, iff its consequence relation is not explosive (principle of explosion: $A, \neg A \models B$, see [16]). The logic is paracomplete, iff there is a set of formulas $\Gamma$ and formulas $A$ and $B$, such that $\Gamma, A \models B$ and $\Gamma, \neg A \models B$, but $\Gamma \not\models B$ (see [1, p. 1092]).

It should be noted that in general under different approaches to matrix logics, the criteria considered are not equivalent.

A logic is called paranormal if it is both paraconsistent and paracomplete. Our research deals with a special class of paranormal logics: literal paranormal logics, that is, logics that simultaneously have paraconsistent and paracomplete
properties at the level of propositional variables and their negations, or, in other words, at the level of literals (see [11, p. 479]).

3. The class of four-valued paranormal logics

In the book [10, p. 56–79] the class of four-valued literal paralogics obtained by combining isomorphs of classical logic found in Bochvar’s four-valued logic \( B_4 \) [3, p. 289] was discussed. As a result, a ten-element upper semilattice\(^2\) is constructed with respect to the functional embedding of matrices that define these literal paralogics and the isomorphs themselves.

For a detailed description of the semilattice, see [10, p. 75–76].

In this study we consider the properties of the class that makes up the supremum of the semilattice in Fig. 1. As it turned out, this class has interesting functional properties; in addition, there are matrices described in the literature that are equivalent in their functional properties to the matrices of the class under consideration. However, there is a question regarding the comparative properties of these matrices with respect to classes of tautologies as well as classes of valid inferences generated by them. These questions are the subject of the proposed study.

Fig. 1. Semilattice of four-valued paralogics

\(^2\)Further in Fig. 1, the sets of basic operations of the corresponding logical matrices are indicated as semilattice elements. The notation is kept only for the supremum of the semilattice and isomorphs; by combining their operations the elements constituting the supremum were obtained.
So, we shall give examples of logical matrices corresponding to paranormal logics that constitute the supremum of the paralogics semilattice [10, p. 69–70].

\[ M_{15} = \langle \{0, 1/3, 2/3, 1\}, \neg_4, \rightarrow_3, \{1, 2/3\} \rangle, \]
\[ M_{16} = \langle \{0, 1/3, 2/3, 1\}, \neg_3, \rightarrow_4, \{1, 1/3\} \rangle. \]

Matrix operations \( \neg_3, \neg_4, \rightarrow_3 \) and \( \rightarrow_4 \) are defined as follows:

| \( x \) | \( \neg_3 x \) | \( \neg_4 x \) | \( \rightarrow_3 \) | \( 1 \) | \( 2/3 \) | \( 1/3 \) | \( 0 \) | \( \rightarrow_4 \) | \( 1 \) | \( 2/3 \) | \( 1/3 \) | \( 0 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0     | 0     | 1     | 1     | 1     | 0     | 0     | 1     | 1     | 0     | 0     |
| 2/3   | 0     | 1     | 2/3   | 1     | 1     | 0     | 0     | 2/3   | 1     | 1     | 1     |
| 1/3   | 1     | 0     | 1/3   | 1     | 1     | 1     | 1     | 1/3   | 1     | 0     | 1     |
| 0     | 1     | 1     | 0     | 1     | 1     | 1     | 1     | 0     | 1     | 1     | 1     |

Matrices
\[ M_3 = \langle \{0, 1/3, 2/3, 1\}, \neg_3, \rightarrow_3, \{1, 2/3\} \rangle \] and
\[ M_4 = \langle \{0, 1/3, 2/3, 1\}, \neg_4, \rightarrow_4, \{1, 1/3\} \rangle \]
correspond to four-valued isomorphs of classical logic, a combination of their matrix operations leads to paranormal systems.

### 3.1. Functional properties

Let us consider the functional properties of logical systems defined by the matrices \( M_{15} \) and \( M_{16} \). Here are some definitions.

Let \( F \) be a set of functions.

**Definition 8.** The closure \([ F ]\) is a set of functions that contains all the superpositions of the functions belonging to \( F \), and only them.

**Definition 9.** The operation of superposition is the operation of generating certain functions through others using formulas.

**Definition 10.** Let \( M = \langle V, F, D \rangle \) and \( M' = \langle V', F', D' \rangle \) be matrices, where \( F \) and \( F' \) are the classes of their basic operations.

\( M' \) is a functional extension of \( M \), if \([ F ] \subseteq [ F' ]\).

\( M \) and \( M' \) are functionally equivalent, if \([ F ] = [ F' ]\).

In [10, p. 78] it is outlined that \( M_{15} \) and \( M_{16} \) are functionally equivalent in the sense indicated above. An explicit proof of this fact follows from the following identities:
(1) \( \neg_4 x = x \rightarrow_4 \neg_3 (x \rightarrow_4 x) \),
\[ x \rightarrow_3 y = \neg_3 y \rightarrow_4 \neg_3 x; \]

(2) \( \neg_3 x = x \rightarrow_3 \neg_4 (x \rightarrow_3 x) \),
\[ x \rightarrow_4 y = \neg_4 y \rightarrow_3 \neg_4 x. \]

Thus, \([\{\neg_4, \rightarrow_3\}] = [\{\neg_3, \rightarrow_4\}]\).

In [2], when constructing his three-valued logic, D.A. Bochvar introduced the concept of an external function.

**Definition 11.** The function \( f \in F \) is said to be external, if \( f(x_1, \ldots, x_n) = 0 \) or \( f(x_1, \ldots, x_n) = 1 \) for any set of truth values \( x_1, \ldots, x_n \).

Let \( B^4_{ex} \) be the set of all external four-valued functions.

Similar to the theorem for three-valued external functions [8, p. 399], there is an analogous one:

**Theorem 1.** Any external function \( F(x_1, \ldots, x_n) \) in \( B^4_{ex} \), that is not identically equal to 0 can be uniquely represented by a perfect disjunctive \( J \)-normal form (\( J \)-pdnf).

The algorithm for constructing \( J \)-pdnf of a four-valued external function is analogous to the one specified by V.K. Finn for the three-valued case [8, p. 399].

We consider a set of all external four-valued functions \( B^4_{ex} \). Let us enumerate a set of all \( n \)-element sets of truth values \( (\alpha^{(j)}_1, \ldots, \alpha^{(j)}_n) \), \( j = 1, 2, \ldots, 4^n \). To each \( j \)-th set we assign a function

\[ K_j(x_1, \ldots, x_n) \equiv J_{\alpha^{(i_1)}_1} x_1 \land J_{\alpha^{(i_2)}_2} x_2 \land \cdots \land J_{\alpha^{(i_n)}_n} x_n. \]

Let \( (\alpha^{(i_1)}_1, \ldots, \alpha^{(i_1)}_n), \ldots, (\alpha^{(i_s)}_1, \ldots, \alpha^{(i_s)}_n) \) be such sets that \( f(\alpha^{(j)}_1, \ldots, \alpha^{(j)}_n) = 1, j = i_1, \ldots, i_s \). Then it is obvious that \( f(x_1, \ldots, x_n) = K_{i_1}(x_1, \ldots, x_n) \lor \cdots \lor K_{i_s}(x_1, \ldots, x_n) \). The function \( f(x_1, \ldots, x_n) = K_{i_1}(x_1, \ldots, x_n) \lor \cdots \lor K_{i_s}(x_1, \ldots, x_n) \) is called \( J \)-pdnf for a four-valued external function.

Here is an example of the construction of \( J \)-pdnf for some arbitrarily chosen external four-valued function.

**Example** Consider a function \( \ast \) from \( B^4_{ex} \) that is defined by the following table:
On properties of a class of four-valued paranormal logics

Then, using the above algorithm for constructing $J$-pdnf we have:

$$x \ast y := (((J_1(x) \land J_1(y)) \lor (J_{2/3}(x) \land J_1(y))) \lor (J_{1/3}(x) \land J_1(y))).$$

Furthermore, the following theorem holds:

**Theorem 2.** $\{\neg_4, \rightarrow_{3}\} = [\mathfrak{B}_{ex}^4]$.

**Proof.** The proof of the theorem follows from:

1. the fact that any external function that is not identically equal to 0 can be uniquely represented in the form $J$-pdnf (Theorem 1);

2. the algorithm to obtain a $J$-pdnf for any external function. That is, to construct $J$-pdnf of some external function by means of functions from a certain set, it is sufficient that this set of functions contains all $J$-operators and functions corresponding to $C$-extending conjunctions and disjunctions$^3$;

3. the following identities, taking into account that $\{\neg_4, \rightarrow_{3}\} = [\{\neg_3, \rightarrow_4\}]$:

$$J_0(x) = \neg_4(\neg_4 x \rightarrow_3 x),$$

$$J_1(x) = \neg_3(x \rightarrow_4 \neg_3 x),$$

$$J_{2/3}(x) = \neg_3(\neg_4 x \rightarrow_3 \neg_3 x),$$

$$J_{1/3}(x) = \neg_3(\neg_3 x \rightarrow_3 \neg_4 x),$$

$$x \lor y = \neg_4 x \rightarrow_3 y,$$

$$x \land y = \neg_4(x \rightarrow_3 \neg_4 y).$$

$^3$The domain of the functions corresponding to such a conjunction and disjunction limited by the subset $\{0, 1\}$ of the set $V_4$, is the classical $\min(x, y)$ and $\max(x, y)$. 

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>$1$</th>
<th>$2/3$</th>
<th>$1/3$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$2/3$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$1/3$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
It is therefore proven that the logical matrices $M_{15}$ and $M_{16}$ from the functional point of view correspond to the class of all external four-valued functions.

The obtained result can be extended to other logical systems whose logical matrices are functionally equivalent to $M_{15}$ ($M_{16}$). As it turns out, there are several such logics described in the literature.

One of them is the logic $V$ [17], which is a formalization of some intuitions of N.A. Vasiliev, which laid the foundation for his imaginary logic. In [17, p. 208] the logical matrix corresponding to $V$:

$M_V = \langle\{0, 1/3, 2/3, 1\}, \neg_4, \to_3, \lor_V, \land_V, \{1\}\rangle,$

where tables for $\neg_4$ and $\to_3$ are given on p. 79, and $\lor_V$ and $\land_V$ are defined as:

\[
\begin{array}{cccc}
\lor_V & 1 & 2/3 & 1/3 & 0 \\
1 & 1 & 1 & 1 & 1 \\
2/3 & 1 & 1 & 1 & 1 \\
1/3 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\land_V & 1 & 2/3 & 1/3 & 0 \\
1 & 1 & 1 & 0 & 0 \\
2/3 & 1 & 1 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

It is clear that $M_{15}$ and $M_V$ are functionally equivalent. In addition, in $M_V$ as initial matrix operations it is sufficient to use $\neg_4$ and $\to_3$, while $\lor_V$ and $\land_V$ could be introduced by definition\(^5\):

\[
x \lor_V y := \neg_3 x \to_3 y,
\]

\[
x \land_V y := \neg_3 (x \to_3 \neg_3 y).
\]

The calculus $V$ [17, p. 207] contains the entire positive fragment of the classical logic $C_2$ plus the following axiom for the negation of $(\sim A \to B) \to ((\sim A \to \sim B) \to A)$ with the restriction on $A$ and $B$: $A$ and $B$ are molecular formulas, that is, they are not atomic (propositional variables). It is proved that if we exclude this restriction, we obtain the axiomatization of classical logic.

In [14] exactly the same matrix appears for the logic $I_0$, a Gentzen-type sequent calculus. In paper [13] a definition of operations embedding the classical propositional logic into $I_0$ is given.

In paper [7] hierarchies of paraconsistent, paracomplete and paranormal logics are presented. The authors characterize these hierarchies by generalizing the so-called (society semantic). In terms of our discussion, of particular interest

\(^4\)Here, for convenience, we use our own notation: the truth value $\top$ from [17] in our notation will be denoted as $2/3$ and the truth value $\bot$ will be denoted as $1/3$.

\(^5\)Taking into account the definition given above: $\neg_4 x := x \to_4 \neg_3 (x \to_4 x)$. 
is a four-valued case of the hierarchy of paranormal logics \( I^n P^k \), where \( I^2 P^2 \) is a logic with the following matrix:

\[
M_{I^2 P^2} = \langle \{0, 1/3, 2/3, 1\}, \neg_4, \rightarrow_3, \{1, 2/3\} \rangle.
\]

Notice that this is exactly the matrix \( M_{15} \).

When it comes to logics \( V \) and \( I_0 \), many authors, in one way or other, establish a connection between these systems and classical propositional logic. Based on the current approach, the connection between these systems and classical logic was clearly demonstrated at the functional level: the matrices \( M_{15} \) and \( M_{16} \) are obtained as a result of combining the isomorphs of the classical logic, and as it was shown above, all these matrices \( M_{15}, M_{16}, M_V, M_I_0 \) and \( M_{I^2 P^2} \) are functionally equivalent and generate the class of all external four-valued functions.

The question arises: If by logic we understand a theory, that is, the class of tautologies defined by the corresponding matrix, how will the systems we are looking at correlate in this case? Therefore, it is sufficient determine whether the matrices \( M_{15}, M_{16} \) and \( M_V \) define the same class of tautologies.

The following section will be devoted to this problem.

### 3.2. Tautology classes

A few things need to be taken into account. Firstly, the matrices \( M_{15} \) and \( M_V \) differ only in the class of designated values (in \( M_{15} D = \{1, 2/3\} \) and in \( M_V D = \{1\} \)) and secondly, the operations in matrices \( M_{15}, M_{16} \) and \( M_V \) are defined by external functions. Granted that, it is clear that in all these matrices a tautology will be a formula with the value 1 under any valuation. Therefore, it is clear that the matrices \( M_{15} \) and \( M_V \) define the same class of tautologies, and it is sufficient to consider the relation of the tautology classes in matrices \( M_{15} \) and \( M_{16} \).

**Theorem 3.** \( E(M_{15}) = E(M_{16}) \).

**Proof.** The proof of the theorem follows from the isomorphism of the matrices \( M_{15} \) and \( M_{16} \) (see [18, Ch.3]). Let us prove an isomorphism of matrices with respect to the mapping \( \phi \).

We define the mapping \( \phi \) as follows:

\[
\phi(0) = 0, \quad \phi(1) = 1, \quad \phi(1/3) = 2/3 \quad \text{and} \quad \phi(2/3) = 1/3.
\]

Further it is easy to verify that

\[
\phi(x \rightarrow_3 y) = \phi(x) \rightarrow_4 \phi(y),
\]

\[
\phi(\neg_4 x) = \neg_3(\phi(x)),
\]
\[ x \in \{1, 2/3\} \text{ iff } \phi(x) \in \{1, 1/3\}. \]

So, the mapping \( \phi \) preserves the matrix operations and the class of designated values of \( D \). Therefore, \( \phi \) is a matrix homomorphism. At the same time, the mapping \( \phi \) is bijective, and any bijective homomorphism is an isomorphism. \( \square \)

Therefore, any logical matrix functionally equivalent to the class of all external four-valued functions, defines one class of tautologies, regardless of the choice of the designated values\(^6\). This class defines a paranormal logic.

### 3.3. Comparison of consequence relations

Let us consider the question of the relation between the logical matrices \( \mathcal{M}_{15}, \mathcal{M}_{16}, \mathcal{M}_V, \mathcal{M}_I \) and \( \mathcal{M}_{I^2P^2} \) with respect the classes of valid inferences generated by them.

From the identity of matrices \( \mathcal{M}_{15} \) and \( \mathcal{M}_{I^2P^2} \) and from the isomorphism of matrices \( \mathcal{M}_{15} \) and \( \mathcal{M}_{16} \) (see the proof of Theorem 3 and [18, Ch.3]) we have:

\[ Cn(\mathcal{M}_{15}) = Cn(\mathcal{M}_{16}) = Cn(\mathcal{M}_{I^2P^2}). \]

Also, since the matrices \( \mathcal{M}_V \) and \( \mathcal{M}_I \) coincide, then

\[ Cn(\mathcal{M}_V) = Cn(\mathcal{M}_I). \]

Next, consider \( Cn(\mathcal{M}_{15}) \) and \( Cn(\mathcal{M}_V) \). It turned out that these two matrices have different properties with respect to classes of valid inferences.

<table>
<thead>
<tr>
<th>( Cn(\mathcal{M}_{15}) )</th>
<th>( Cn(\mathcal{M}_V) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p, p \rightarrow_3 q \models q )</td>
<td>( p, p \rightarrow_3 q \nvdash q )</td>
</tr>
<tr>
<td>( p, \neg_4p \nvdash q )</td>
<td>( p, \neg_4p \models q )</td>
</tr>
<tr>
<td>( q \nvdash p, \neg_4p )</td>
<td>( q \nvdash p, \neg_4p )</td>
</tr>
</tbody>
</table>

As previously stated on p. 77, different criteria for paraconsistency and paracompleteness can be used. So, if we consider logic in terms of consequence relations, the paraproperties are defined in terms of explosiveness and implosiveness. It should be noted that in the matrix \( \mathcal{M}_V \) the consequence relation is explosive, while in \( \mathcal{M}_{15} \) it is not. So, the question arises about the relationship between the various criteria for paraproperties. For example, for paraconsistency, the deductive criterion is stronger and involves satisfying an “implicative-negative” criterion, while the opposite is not necessary (as can be

\(^6\)Moreover, if the matrices in question are constructed for different propositional languages, then due to the functional equivalence of the matrices, there is a one-to-one correspondence between the classes of their tautologies.
seen in the case of the matrix $M_V$). In [17], the logic $V$ is considered as a theory, therefore the properties of consequence relation are not taken into account.

Let us look at some other properties of the matrix $M_V$. We can see that the rule *modus ponens* does not preserve the designated values\(^7\), that is, the matrix $M_V$ is not normal in the sense of Łukasiewicz–Tarski [12, p. 134]. In this case it is sufficient that the rule *modus ponens* preserves tautologies.

However, it should be noted that the difference in the classes of valid inferences generated by the matrices $M_{15}$ and $M_V$ takes place only at the propositional (literal) level, since in this case it is important to choose the class of designated values. And conversely, as in both matrices the same operations are used and they are defined by external functions, then in the case of molecular formulas all the differences are gone. So, at the molecular level the consequence relation in both matrices will be explosive, that is, $A, \neg A \models B$ (the formula $A$ is not a propositional variable), since it is not possible that $v(A) \in D$ and $v(\neg A) \in D$.

From the deductive point of view, the negation $\neg 4$ has different properties in $M_{15}$ and $M_V$. Here it is essential to choose the class of designated values. So, in $M_{15}$ the negation $\neg 4$ is paraconsistent at the propositional level, as it does not form contradictions, that is, there exists a value when $v(p) \in D$ and $v(\neg p) \in D$, while it is not paraconsistent in $M_V$; as a consequence of this in $M_V$ consequence relation is explosive. However, because the matrix $M_V$ is not normal in the Łukasiewicz–Tarski sense, nothing can be deduced from the contradiction by the rule *modus ponens*.

4. **Conclusion**

We considered the class of four-valued paranormal logics that are functionally equivalent to each other and represent the class of all four-valued external functions. Some of the logics from this class were constructed by different authors based on different motivations. In one way or another, the question of the relation of these systems to classical logic was considered. In our study [10, § 3.1] we obtained two paranormal logical matrices as a result of combining the isomorphs of classical logic found in Bochvar’s four-valued logic $B_4$. Therefore, the connection between these systems and classical logic is visible at the functional level.

Furthermore, as a result of studying the properties of this class of four-valued literal paranormal logics, it is shown that all systems are equivalent with respect to the classes of tautologies. Therefore, different logical matrices specify the same paranormal theory. The study of the consequence relation properties

---

\(^7\)For details on distinguishing the formulations of the rule *modus ponens* regarding the preservation of designated values and preservation of tautologies, see [5, p. 101].
in the paranormal matrices in question showed deductive differences between them. There are two groups of matrices: $C_n(M_{15}) = C_n(M_{16}) = C_n(M_{15})$ and $C_n(M_V) = C_n(M_{16})$. It was shown that in the second group of matrices the consequence relation is explosive, but since the rule of modus ponens does not preserve the designated values, nothing can be deduced from the contradiction.

References


On properties of a class of four-valued paranormal logics


О свойствах одного класса четырехзначных параполных логик

Статья посвящена результатам, полученным в ходе исследования одного класса четырехзначных литеральных параполных логик, т. е. логик, которые одновременно являются паранепротиворечивыми и параполными на уровне пропозициональных переменных и их отрицаний. Паранепротиворечивые логики допускают возможность работы с противоречивой информацией, параполные логики позволяют строить рассуждения в условиях неполноты информации. С обоими типами неопределенности, как с противоречивой, так и с неполной информацией, позволяют работать парапараллельные системы.

В [5] рассмотрен класс четырехзначных литературных параполлических, полученных методом комбинирования изоморфов классической логики, выделенных в четырехзначной логике Бочвара В4. В результате вместе с самими изоморфами логические матрицы, определяющие эти логики, образуют десятиэлементную верхнюю полурешетку относительно функционального вложения. В предложенной статье мы исследуем класс матриц, состоящий из супремума упомянутой полурешетки. Как оказалось, матрицы этого класса обладают интересными функциональными свойствами, а именно соответствуют классу всех внешних четырехзначных функций. В статье также проводится алгоритм построения совершенной дизъюнктивной J-нормальной формы четырехзначной внешней функции. В литературе имеются известные матрицы, которые функционально эквивалентны матрицам рассматривающего класса. Например, одна из них это матрица, определяющая логику V [17], представляющая собой формализацию интуиций воображаемой логики Н.А. Васильева. Нами рассмотрен вопрос о соотношении всех этих систем как по классам тавтологий, так и по классам правильных заключений, порождаемых рассматриваемыми матрицами. В результате доказано, что по классу тавтологий все системы эквивалентны, однако отличаются по свойствам отношения логического следования.

Ключевые слова: четырехзначные параполные логики, функциональные свойства, внешние функции, тавтологии, отношение следования

Литература

О свойствах одного класса четырёхзначных паранормальных логик


