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Deductive systems with unified multiple-conclusion rules

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Abstract: Our goal is to develop a syntactical apparatus for propositional logics in which the accepted and rejected propositions have the same status and are being treated in the same way. The suggested approach is based on the ideas of Łukasiewicz used for the classical logic and in addition, it includes the use of multiple conclusion rules. More precisely, a consequence relation is defined on a set of statements of forms “proposition A is accepted” and “proposition A is rejected”, where A is a proposition, — unified consequence relation. Accordingly, the rules defining a unified consequence relation have statements as premises, and conclusions — unified rules. A special attention is paid to the logics in which each proposition is either accepted or rejected. If we express this property via unified rules and add them to a unified deductive system, such a unified deductive system defines a reversible unified consequence: a statement “proposition B is accepted” is derived from “proposition A is accepted” if and only if a statement “proposition A is rejected” is derived from “proposition B is rejected”.

Keywords: propositional logic, logic with refutation, deductive system with rejection, multiple conclusion rules

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1. Introduction

Our goal is to develop a framework for the logics that accommodate rejected propositions along with asserted propositions. We call such logics *unified*¹. And we abolish the regular implicit assumption that every proposition which is not asserted is rejected.

¹Sometimes logics with refutations are called “hybrid” (e.g. [Goranko, 2019]); we prefer to call such logics “unified”, because the term “hybrid logic” is often used in a different sense (e.g. [Torben, 2011]).

Thus, we need to answer the following questions:

- (a) What do we prove?
- (b) How do we prove it?

The brief answers are:

- (a) We prove statements asserting or rejecting a given proposition;
- (b) We use the multiple-conclusion rules in which the premises and conclusions are finite sets of such statements.

It is due to Łukasiewicz that rejection was explicitly included in logic. In the introduction to his 1921 paper translated in [Łukasiewicz, Jan, 1970], he wrote: “The concepts of ‘truth’, ‘falsehood’, and ‘assertion’ I owe to Frege. In adding ‘rejection’ to ‘assertion’ I have followed Brentano”.

According to Brentano and in contrast to Frege, assertion (or acceptance) and rejection (or refutation, or denial) should have the same status. Let us note that assertion of a negation is much stronger than the rejection (cf. [Restall, 2015]). For instance, in the Classical Logic we reject formula p , where p is a propositional variable (in symbols $\neg p$), but the assertion $\vdash \neg p$ does not hold.

In particular, Łukasiewicz suggested to endow ordinary calculus (with rule of substitution) defining the Classical Logic (CPC), with the anti-axiom $\neg p$ and the following two rules:

$$\begin{array}{ll} \text{modus tollens:} & \vdash (A \rightarrow B), \neg B / \neg A \quad (\text{MT}) \\ \text{reversed substitution:} & \neg \sigma(A) / \neg A \quad (\text{RS}) \end{array}$$

Independently, in [Carnap, 1942] Carnap suggested to include the rejections into deductive systems: “The rules of deduction usually consist of primitive sentences and rules of inference (defining ‘directly definable in K’). Sometimes, K contains also rules of refutation (defining ‘directly refutable in K’).” Moreover, Carnap also used the multiple-conclusion rules.

Carnap’s goal was to achieve the categoricity: if we attempt to axiomatize a two-valued logic, the only (up to isomorphism) model for this logic must be a two-element model.

Our goal is not to limit the class of models. We follow the Brentano-Łukasiewicz path.

Our approach is similar to the approach accepted in [Smiley, 1996] or [Rumfitt, 2000]: we consider the signed propositions and we develop the machinery needed for the derivations of a signed proposition from the signed propositions. This “sign” is different from the truth operator (cf. [von Wright, 1987; Pavlov,

2004; Pavlov, 2011]) because we do not allow the mixed formulas (cf. [von Wright, 1987]): the signed formulas are entities of a metalanguage. For the same reason, the signed formulas are also different from the formulas of Bochvar's logic (cf. [Bochvar, 1939]): we do not allow the signs (which are elements of a metalanguage) to intermingle with the connectives (which are elements of language), and we do not allow the nested signs. The signed formulas can also be viewed as judgments (cf. [Kracht, 2010].)

In the present paper, we further develop the approach introduced in [Citkin, 2015]. In contrast to the aforementioned paper, here we do not consider multiple-conclusion logics in the sense of [Shoesmith and Smiley, 2008] or Carnap's junctives [Carnap, 1943]. We use multiple-conclusion rules merely as the means of derivation of a statement from a set of statements. Additionally, we are focusing on logics in which the laws of excluded middle and non-contradiction are expressed as statements.

From the very general standpoint², there are two ways of how to handle refutation syntactically: direct and indirect. To determine whether formula A is refutable one can do one of the two things:

- (a) to derive a statement about refutability of A in a meta-logic ($\mathbb{L}$ -derivation — Łukasiewicz-style derivation)
- (b) to derive a formula B that we already know is refuted from A , and apply Modus Tollens (C-derivation — indirect derivation, Carnap's way).

Example 1. Let us consider the classical propositional calculus and compare $\mathbb{L}$ - and C-derivation of the refutability of the proposition $\neg p$ if we accepted that proposition p is rejected:

L-derivation		C-derivation	
$\neg p$	anti-axiom	$\neg p$	assumption
$\vdash \neg(p \rightarrow p) \rightarrow p$	tautology	$\neg(p \rightarrow p)$	Sb
$\vdash \neg(p \rightarrow p)$	MT	$\neg(p \rightarrow p) \rightarrow p$	tautology
$\vdash \neg p$	RS	p	MP
		reject $\neg p$	because p is rejected

Remark 1. Let us observe that in C-derivations Modus Tollens is not the same as in $\mathbb{L}$ -derivations:

$$\begin{array}{cc}
 \text{L-derivations} & \text{C-derivations} \\
 \vdash (A \rightarrow B), \neg B / \neg A & A \vdash B, \neg B / \neg A
 \end{array}$$

²More details about different approaches to refutation systems the reader can find in [Goranko et al., 2020; Goranko, 2019; Wybraniec-Skardowska, 2018; Citkin, 2015].

In L-derivations, Modus Tollens is a rule of logic, while in C-derivations, Modus Tollens is a rule of meta-logic. In terms of [Scott, 1974], the first version of Modus Tollens is about connective of implication, while the second one is about relationship of implication.

An existence of an L-derivation entails the existence of C-derivation. The converse is true only under some assumptions, more precisely, a weak form of the deduction theorem (cf. [Staszek, 1971]).

The paper is structured as follows. In Section 2, we give the definition of unified logics and we consider some of their properties. In Section 3, we introduce the unified deductive systems — a syntactic way of defining unified logics. In Section 4, we study the standard unified deductive systems — the unified deductive systems in which the rules representing the laws of excluded middle and non-contradiction are derivable.

2. Unified logics

In this section we introduce the notion of unified logic.

2.1. Basic definitions

We consider a propositional language consisting of a countable set of propositional variables P and a finite set of connectives (not including signs $\oplus, \ominus, \odot, \blacktriangledown, \blacktriangle, \vdash$ which are reserved for use in the meta-logic). By formula (or proposition) we understand a propositional formula in this language defined in a regular way. Frm denotes the set of all formulas.

It is customary to define logic as a consequence relation. If assertions and rejections have the same status, we need to consider the consequence relations on sets of (*meta*-)statements of the type “ A being asserted” and “ A being rejected”, and we use $\oplus A$ for the former and $\ominus A$ for the latter. $\blacktriangledown$ and $\blacktriangle$ are two special statements the role of which will become apparent later. Loosely speaking, one can think of $\blacktriangledown$ as a disjunction of an empty set of statements and thus, being always accepted, while $\blacktriangle$ being a conjunction of an empty set of statements and thus, being always rejected.

The statements of the form $\oplus A$ and $\blacktriangledown$ are *positive*, the statements of the form $\ominus A$ and $\blacktriangle$ are *negative*. If α is a statement, $\bar{\alpha}$ denotes a *reversed statement*:

$$\bar{\alpha} = \begin{cases} \ominus A, & \text{if } \alpha = \oplus A; \\ \oplus A, & \text{if } \alpha = \ominus A, \end{cases}$$

and $\bar{\blacktriangledown} = \blacktriangle$, and $\bar{\blacktriangle} = \blacktriangledown$. The statements of form $\oplus A$ and $\ominus A$, where A is a proposition, are *proper*.

If X is a set of propositions, we let

$$X^+ := \{\oplus A \mid A \in X\} \text{ and } X^- := \{\ominus A \mid A \in X\}.$$

Let $\mathcal{S}$ denote a set of all statements. If Γ is a set of statements, Γ^+ and Γ^- denote respectively the subsets of positive and negative statements from Γ .

It is inconvenient for our purposes to use Łukasiewicz's notations $\vdash$ and $\dashv$ for “being asserted” and “being rejected”, because the expression like

$$\vdash A_1, \dots, \vdash A_n \vdash \vdash B$$

looks confusing. The notation³ $\oplus A$ and $\ominus A$ eliminates the confusion:

$$\oplus A_1, \dots, \oplus A_n \vdash \oplus B.$$

Definition 1. A *unified logic* is a relation $\vdash$ on $2^{\mathcal{S}} \times \mathcal{S}$ satisfying the following conditions: for any non-empty $\Gamma \subseteq \mathcal{S}$, any $\Delta \subseteq \mathcal{S}$, and any statements α, β ,

- (R) $\Gamma, \alpha \vdash \alpha$;
- (M) $\Gamma \vdash \alpha$ entails $\Gamma \cup \Delta \vdash \alpha$;
- (T) $\Gamma \vdash \alpha$ and $\alpha, \Delta \vdash \beta$ entails $\Gamma, \Delta \vdash \beta$;
- (C) $\alpha \vdash \blacktriangledown$ and $\blacktriangle \vdash \alpha$;
- (P) $\Gamma, \blacktriangledown \vdash \alpha$ entails $\Gamma \vdash \alpha$.

A unified logic $\vdash$ is *finitary* if $\Gamma \vdash \alpha$ yields that there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \alpha$.

Immediately from (T) and (C) it follows that for any $\Gamma \subseteq \mathcal{S}$, if $\Gamma \vdash \blacktriangle$, then $\Gamma \vdash \beta$ for every statement β .

If σ is a substitution, that is, $\sigma : \mathcal{P} \rightarrow \text{Frm}$, then, in a natural way, one can extend σ to statements: $\sigma(\blacktriangledown) = \blacktriangledown$, $\sigma(\blacktriangle) = \blacktriangle$ and

$$\sigma(\odot A) = \odot \sigma(A),$$

where $\odot \in \{\oplus, \ominus\}$.

Each unified consequence relation $\vdash$ defines a set of *asserting* (or *positive*) *theses*:

$$\text{Th}_{\vdash}^+ := \{\alpha \in \mathcal{S}^+ \mid \blacktriangledown \vdash \alpha\}$$

and a set of *refuting theses* (or *negative*, or *anti-theses*):

$$\text{Th}_{\vdash}^- := \{\alpha \in \mathcal{S}^- \mid \blacktriangledown \vdash \alpha\}.$$

The set $\text{Th}_{\vdash} := \text{Th}_{\vdash}^+ \cup \text{Th}_{\vdash}^-$ is a set of *theses*. Let us note that $\blacktriangledown \in \text{Th}_{\vdash}^+ \subseteq \text{Th}_{\vdash}$.

³The similar notations are used in [Smiley, 1996] and [Rumfitt, 2000].

Proposition 1. Let $\vdash$ be a unified logic and $\alpha \in \mathcal{S}$. If $\alpha \in \text{Th}_\vdash$, then $\Gamma \vdash \alpha$ for any $\Gamma \subseteq \mathcal{S}$.

Proof. If $\alpha \in \text{Th}_\vdash$, then $\nabla \vdash \alpha$ and by (M), we have $\Gamma, \nabla \vdash \alpha$. If $\Gamma = \emptyset$, the statement is trivial. Otherwise, we can apply (P) and obtain $\Gamma \vdash \alpha$. ■

Note, that $\text{Th}_\vdash^+ \cap \text{Th}_\vdash^- = \emptyset$ simply because $\mathcal{S}^+ \cap \mathcal{S}^- = \emptyset$ by a purely syntactical reason. But the situation is different if we consider the sets of formulas:

$$\begin{aligned} \text{L}_\vdash^+ &:= \{A \in \text{Frm} \mid \oplus A \in \text{Th}_\vdash\} \quad \text{is a set of } \textit{theorems}; \\ \text{L}_\vdash^- &:= \{A \in \text{Frm} \mid \ominus A \in \text{Th}_\vdash\} \quad \text{is a set of } \textit{anti-theorems}. \end{aligned}$$

And it is possible that $\text{L}_\vdash^+ \cap \text{L}_\vdash^- \neq \emptyset$ and $\text{L}_\vdash^+ \cup \text{L}_\vdash^- \neq \text{Frm}$.

Definition 2. Let $\vdash$ be a unified logic.

$$\begin{aligned} \text{If } \text{L}_\vdash^+ \cap \text{L}_\vdash^- &= \emptyset, & \text{the logic is } \textit{coherent}, \\ \text{if } \text{L}_\vdash^+ \cup \text{L}_\vdash^- &= \text{Frm}, & \text{the logic is } \textit{full}. \end{aligned}$$

A full and coherent logic is called *conventional*. And $\vdash$ is *trivial* if $\text{L}_\vdash^+ = \text{L}_\vdash^- = \text{Frm}$.

Clearly, a unified logic $\vdash$ is conventional if and only if $\text{L}^- = \text{Frm} \setminus \text{L}^+$.

Proposition 2. A unified logic $\vdash$ is coherent if and only if

$$\alpha \in \text{Th}_\vdash \text{ entails } \bar{\alpha} \notin \text{Th}_\vdash \text{ (that is, } \text{Th}_\vdash \cap \overline{\text{Th}_\vdash} = \emptyset),$$

and $\vdash$ is full if and only if for any statement α ,

$$\alpha \in \text{Th}_\vdash \text{ or } \bar{\alpha} \in \text{Th}_\vdash \text{ (that is, } \text{Th}_\vdash \cup \overline{\text{Th}_\vdash} = \mathcal{S}).$$

Proof. Straightforward. ■

Corollary 1. A unified logic $\vdash$ is conventional if and only if $\overline{\text{Th}_\vdash} = \mathcal{S} \setminus \text{Th}_\vdash$.

3. Unified deductive systems

In this section, we will see how a unified logic can be defined by a unified deductive system. As usual, a unified deductive system is an ordered pair $\langle \text{Ax}; \text{R} \rangle$, where $\text{Ax} \subseteq \mathcal{S}$ is a set of axioms, and R is a set of unified rules. First, let us clarify what is a unified rule and then, in Section 3.4, we define the unified inference.

3.1. Unified inference rules

If Γ, Δ are finite **non-empty** sets of statements, an ordered pair Γ/Δ (or $\frac{\Gamma}{\Delta}$) is called a *structural multiple-conclusion* or *multiple-alternative* rule (m-rule for short). The premises Γ are viewed conjunctively, while the alternatives (conclusions) Δ are viewed disjunctively.

In rules, ∇ and $\blacktriangle$ respectively play the role of empty sets of premises and alternatives: the notation $\nabla/\blacktriangle$ is certainly less misleading than $\emptyset/\emptyset$, or $-$.

We divide m-rules into two categories: if $r := \Gamma/\Delta$ is an m-rule, then

- r is *conclusive* if Δ consists of a single statement;
- if $\Delta = \{\blacktriangle\}$, the rule is also *terminating*;
- r is *inconclusive* if Δ consists of more than one statement.

For instance, the rule $\oplus p, \oplus(p \rightarrow q)/\oplus q$ is conclusive; the rule $\oplus(p \vee q)/\oplus p, \oplus q$ is inconclusive; the rule $\oplus p, \ominus p/\blacktriangle$ is terminating.

Note that **Sb** and **RS** are not m-rules. If R is a set of m-rules, we let $R^s = R \cup \{\mathbf{Sb}, \mathbf{RS}\}$.

Multiple-alternative rules allow to explicitly use the proofs by cases. Indeed, in the setting of natural deduction, a proof by cases looks like this:

$$\frac{\begin{array}{ccc} [A] & & [B] \\ & \vdots & \vdots \\ A \vee B & C & C \end{array}}{C}$$

In the multiple-alternative setting, a proof by cases looks more straightforward:

$$\frac{\begin{array}{c} A \vee B \\ \vdots \\ \begin{array}{|c|c|} \hline A & B \\ \hline \end{array} \\ \vdots \\ C \end{array}}{(p \vee q)/p, q}$$

By applying rule $(p \vee q)/p, q$, we get the alternatives p and q to be considered separately.

3.2. Admissible m-rules

An m-rule Γ/Δ is *admissible* for a given unified logic $\vdash$, if for each substitution σ ,

$$\sigma(\Gamma) \subseteq \text{Th}_{\vdash} \implies \sigma(\Delta) \cap \text{Th}_{\vdash} \neq \emptyset.$$

For instance, a rule $\Gamma/\blacktriangle$ is admissible for a unified logic $\vdash$ if and only if there is no such a substitution σ that $\sigma(\Gamma) \subseteq \text{Th}_{\vdash}$. In particular, the rule $\nabla/\blacktriangle$ is not admissible for any nontrivial unified logic.

By *unified intermediate logic* we understand a unified logic the set of positive theorems of which extends the set of theorems of the intuitionistic propositional logic (IPC) and is included in the set of the theorems of classical propositional logic (CPC), and the set of anti-theorems coincides with the set of non-theorems.

Proposition 3. For any unified intermediate logic $\vdash$ and for any formula A , rule $\oplus A/\blacktriangle$ is admissible if and only if rule $\nabla/\oplus \neg A$ is admissible.

Proof. Indeed, if $\nabla/\oplus \neg A$ is admissible for $\vdash$, then, $\neg A$ is a theorem of $\vdash$ and hence, no substitution makes A a theorem. Thus, $\oplus A/\blacktriangle$ is admissible for $\vdash$.

Conversely, suppose that $\oplus A/\blacktriangle$ is admissible for $\vdash$. Then, there is no substitution σ such that $\sigma(A)$ is a theorem of $\vdash$. In particular, for any substitution $\sigma : \mathbf{P} \rightarrow \{p \rightarrow p, p \wedge \neg p\}$, $\sigma(A)$ is not a theorem of $\vdash$, which means that for any valuation ν in a two-element Boolean algebra, $\nu(A) = 0$ and hence, $\nu(\neg A) = \neg \nu(A) = 1$. Therefore, $\neg A$ is a Boolean tautology and consequently, $\neg A$ is a theorem of CPC. Then, by the Glivenko Theorem, $\neg A$ is a theorem of IPC and hence, $\neg A$ is a theorem of $\vdash$, that is, $\nabla/\oplus \neg A$ is admissible for $\vdash$. ■

The following proposition is an immediate consequence of the definitions.

Proposition 4. Let $\vdash$ be a unified logic. Then, $\vdash$ is coherent if and only if the following rule is admissible for it:

$$\text{Co} := \frac{\oplus p, \ominus p}{\blacktriangle}. \quad (\text{Coherency rule})$$

And $\vdash$ is full if and only if the following rule is admissible for it:

$$\text{Fu} := \frac{\nabla}{\oplus p, \ominus p}. \quad (\text{Fullness rule})$$

In what follows, the above m-rules play the central role.

Proposition 5. Let $\vdash$ be a conventional unified logic (that is, $\vdash$ is full and coherent). Then, for any finite sets Γ, Δ and any proper statement α ,

if the rule $\frac{\alpha, \Gamma}{\Delta}$ is admissible for $\vdash$, then the rule $\frac{\Gamma}{\overline{\alpha}, \Delta}$ is admissible too;

if the rule $\frac{\Gamma}{\alpha, \Delta}$ is admissible for $\vdash$, then the rule $\frac{\overline{\alpha}, \Gamma}{\Delta}$ is admissible too.

If $\Gamma = \emptyset$, then $\Gamma/\overline{\alpha}, \Delta$ is understood as $\nabla/\alpha, \Delta$; and if $\Delta = \emptyset$, then $\overline{\alpha}, \Gamma/\Delta$ is understood as $\overline{\alpha}, \Gamma/\blacktriangle$.

In other words, without violating admissibility, one can move a statement from the premises to the alternatives, or vice-versa, with changing the “sign” of the statement.

Proof. Suppose that $\Gamma \neq \emptyset$ and rule $\alpha, \Gamma / \Delta$ is admissible for a unified logic $\vdash$. Then, for any substitution σ such that $\sigma(\Gamma) \subseteq \text{Th}_\vdash$, because $\vdash$ is conventional, by Corollary 1, either $\sigma(\alpha) \in \text{Th}_\vdash$, or $\sigma(\bar{\alpha}) \in \text{Th}_\vdash$. In the former case, $\alpha, \Gamma / \Delta$ is admissible, $\sigma(\Delta) \cap \text{Th}_\vdash \neq \emptyset$ and hence, $\sigma(\bar{\alpha}, \Delta) \cap \text{Th}_\vdash \neq \emptyset$. In the latter case, obviously, $\sigma(\bar{\alpha}, \Delta) \cap \text{Th}_\vdash \neq \emptyset$. Thus, the rule $\bar{\alpha}, \Gamma / \Delta$ is admissible for $\vdash$.

Suppose that rule α / Δ is admissible. Then, because $\vdash$ is conventional, for every substitution σ , either $\sigma(\alpha) \in \text{Th}_\vdash$, or $\sigma(\bar{\alpha}) \in \text{Th}_\vdash$. In the former case, because rule α / Δ is admissible, $\sigma(\Delta) \cap \text{Th}_\vdash \neq \emptyset$ and consequently, $\sigma(\bar{\alpha}, \Delta) \cap \text{Th}_\vdash \neq \emptyset$. In the latter case, $\sigma(\bar{\alpha}) \in \text{Th}_\vdash$ and consequently, $\sigma(\bar{\alpha}, \Delta) \cap \text{Th}_\vdash \neq \emptyset$. Thus, $\sigma(\bar{\alpha}, \Delta) \cap \text{Th}_\vdash \neq \emptyset$ for any substitution σ and therefore, the rule $\nabla / \bar{\alpha}, \Delta$ is admissible.

The rest of the cases can be proved with a similar argument. ■

Example 2. Let $\vdash$ be a unified conventional logic signature of which contains $\rightarrow$. If Modus Ponens is admissible for $\vdash$, then, all the following eight variations of Modus Ponens are admissible:

$$\begin{array}{c} \frac{\nabla}{\oplus p, \oplus(p \rightarrow q), \oplus q}; \quad \frac{\oplus p}{\oplus(p \rightarrow q), \oplus q}; \quad \frac{\oplus(p \rightarrow q)}{\oplus p, \oplus q}; \quad \frac{\ominus q}{\oplus p, \oplus(p \rightarrow q)}; \\[10pt] \frac{\oplus p, \oplus(p \rightarrow q)}{\oplus q}; \quad \frac{\oplus p, \ominus q}{\ominus(p \rightarrow q)}; \quad \frac{\oplus(p \rightarrow q), \ominus q}{\ominus p}; \quad \frac{\oplus p, \oplus(p \rightarrow q), \ominus q}{\blacktriangle}. \end{array}$$

Let us note that for conventional logics, admissibility of Modus Ponens entails the admissibility of Modus Tollens.

By the same argument, for the rule of substitution we have two variations which are either simultaneously admissible, or simultaneously not admissible:

$$\frac{\oplus A}{\oplus \sigma(A)}; \quad \frac{\ominus \sigma(A)}{\ominus A}.$$

3.3. Protoinference

Let R be a set of rules (which may include **Sb** and/or **RS**) and Γ be a non-empty set of statements. A *protoinference from Γ by R* (or $\langle \Gamma; R \rangle$ -*protoinference* for short) is a finite directed tree, the nodes of which are labeled by statements, and it is defined by induction⁴.

A leaf of protoinference labeled by $\blacktriangle$ is *terminating* (we have reduced a case to contradiction), otherwise, it is *extendable*.

Like in a Hilbert-style inference, in protoinference we use the assumptions and apply the inference rules.

⁴Note that we define a protoinference from $\langle \Gamma; R \rangle$ without clarifying what we are deriving.

(a) A tree consisting of a single node (a root) labeled by $\blacktriangledown$ is a $\langle \Gamma; \mathbf{R} \rangle$ -protoinference;

(b) *using the assumptions*: if $\mathfrak{I}$ is a $\langle \Gamma; \mathbf{R} \rangle$ -protoinference, then any extendable leaf can be extended by adjoining a leaf labeled by a statement from Γ , and the obtained tree is a $(\Gamma, \mathbf{R})$ -protoinference;

(c) *applying the rules*: if $\mathfrak{I}$ is a $\langle \Gamma; \mathbf{R} \rangle$ -protoinference, then any extendable leaf λ can be extended by adjoining the leaves labeled by statements from a finite set Δ , provided there is an instance Ξ/Δ of a rule from $\mathbf{R}$, and each statement from Ξ is labeling a node between λ and the root (see Fig. 1).

The trees obtained in such a way, and only them, are the $(\Gamma, \mathbf{R})$ -protoinferences.

Let us observe that the rules of form $\blacktriangle, \Gamma/\Delta$ cannot be used in any protoinference, because $\blacktriangle$ labels only a terminating leaf.

Fig. 1. Application of a rule

If there is a $\langle \Gamma; \mathbf{R} \rangle$ -protoinference the leaves of which are labeled by statements only from a finite set of statements Δ , we denote this by $\Gamma \sim_{\mathbf{R}} \Delta$.

Let us note that immediately from the definition of protoinference, for any finite sets of statements $\Gamma, \Gamma', \Delta, \Delta'$,

$$\Gamma \sim_{\mathbf{R}} \Delta \quad \text{entails} \quad \Gamma \cup \Gamma' \sim_{\mathbf{R}} \Delta \cup \Delta'. \quad (1)$$

Proposition 6. Suppose that Γ, Δ are finite sets of statements and R is a set of rules. Then the following holds:

- (a) $\nabla, \Gamma \vdash_R \Delta$ if and only if $\Gamma \vdash_R \Delta$;
 - (b) $\Gamma \vdash_R \blacktriangle, \Delta$ if and only if $\Gamma \vdash_R \Delta$;
 - (c) $\Gamma \vdash_R \nabla, \Delta$;
 - (d) $\blacktriangle, \Gamma \vdash_R \Delta$.
- (2)

Proof. (a) follows from the observation that by the definition of protoinference, any $\langle \nabla, \Gamma; R \rangle$ -protoinference of Δ is at the same time a $\langle \Gamma; R \rangle$ -protoinference of Δ , and vice-versa.

(b) Because by definition the extendable leaves do not contain $\blacktriangle$, the definition of protoinference entails that every $\langle \Gamma; R \rangle$ -protoinference of $\blacktriangle, \Delta$ is at the same time, a $\langle \Gamma; R \rangle$ -protoinference of Δ , and vice-versa: in both cases, extendable leaves contain statements only from Δ .

(c) A tree consisting only of a root labeled by ∇ is a protoinference, that is, $\Gamma \vdash_R \nabla, \Delta$.

(d) A tree consisting of two nodes: a root labeled by ∇ , and a leaf labeled by $\blacktriangle$, is a $\langle \blacktriangle; R \rangle$ -protoinference of Δ , that is, $\blacktriangle, \Gamma \vdash_R \Delta$. ■

3.4. Unified inference

Suppose that Γ, Δ are nonempty sets of statements, α is a statement and R is a set of rules.

Definition 3. A $\langle \Gamma; R \rangle$ -protoinference is a $\langle \Gamma; R \rangle$ -inference provided all its extendable leaves contain the same statement.

Definition 4. A statement α is derivable from statements Δ by $\langle \Gamma; R \rangle$, if there is a $\langle \Delta \cup \Gamma; R \rangle$ -inference all extendable leaves of which are labeled by α , that is, if $\Delta \cup \Gamma \vdash_R \alpha$.

Note that if a protoinference does not contain any extendable leaves, that is, all its leaves are labeled by $\blacktriangle$, this protoinference is an inference of any statement.

Roughly speaking, α is derivable from Δ if after we have considered every case arisen in the inference, we either have derived α , or we have arrived at a contradiction.

Proposition 7. Any pair $\mathcal{D} = \langle \Gamma; R \rangle$ consisting of a nonempty set of statements Γ and a set of rules R , defines a unified logic:

$$\Delta \vdash_{\mathcal{D}} \alpha \quad \Leftrightarrow \quad \alpha \text{ is derivable from } \Delta \text{ by } \langle \Gamma; R \rangle.$$

Proof. Properties (R),(M),(C),(P) from Definition 1 are evident.

To prove (T), assume that $\mathfrak{I}$ is an inference of α from Γ , and $\mathfrak{I}'$ is an inference of β from α, Δ . Without loss of generality, we can assume that in $\mathfrak{I}'$ the root labeled by $\blacktriangledown$ has a unique successor labeled by α . Let $\mathfrak{I}'_\alpha$ be a tree obtained from $\mathfrak{I}'$ by removing the root. It is clear that $\mathfrak{I}'_\alpha$ is a rooted labeled tree and its root is labeled by α .

Now we can construct a desired inference $\mathfrak{I} + \mathfrak{I}'$ of β from Γ, Δ by identifying the leaves of $\mathfrak{I}$ labeled by α with the root of $\mathfrak{I}'_\alpha$ (Fig. 2). ■

Fig. 2. Transitivity

4. Standard Unified Deductive Systems

Usually logics are defined either by semantic means — matrices, algebras, Kripke models, etc. — or by syntactic means, the deductive systems. Now we are on a position to define the unified deductive systems.

4.1. Unified deductive systems

Definition 5. Unified deductive system is an ordered pair $\langle Ax; R^s \rangle$, where Γ is a nonempty set of statements, and R is a set of m-rules. The positive statements from Ax are *axioms*, while the negative statements from Γ are *anti-axioms*.

Example 3. Łukasiewicz's logic with refutation can be defined by the following unified deductive system $\mathcal{C}$:

$$\begin{array}{ll}
 \text{axioms:} & \{\oplus p \rightarrow (q \rightarrow p), \\
 & \oplus(p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))), \\
 & \oplus((\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p))\} \\
 \text{anti-axiom:} & \{\ominus p\} \\
 \text{rules:} & \{\text{MP, MT}\} \cup \{\text{Sb, RS}\}
 \end{array}$$

By Proposition 7, every unified deductive system $\mathcal{D} = \langle \text{Ax}; \text{R}^s \rangle$ defines a unified logic $\vdash_{\mathcal{D}}$, the theses of which we denote by $\text{Th}_{\mathcal{D}}^+$ and $\text{Th}_{\mathcal{D}}^-$, while the sets of theorems and anti-theorems are denoted by $\text{L}_{\mathcal{D}}^+$ and $\text{L}_{\mathcal{D}}^-$.

Let us observe that the set $\text{Th}_{\mathcal{D}}^+$ and consequently, $\text{L}_{\mathcal{D}}^+$ is closed with regard to **Sb**, and the set $\text{Th}_{\mathcal{D}}^-$ and consequently, $\text{L}_{\mathcal{D}}^-$ is closed relative to **RS**.

Let $\text{L}^+, \text{L}^- \subseteq \text{Frm}$. An ordered pair $\text{L} = (\text{L}^+, \text{L}^-)$ is *substitutionally closed* (s-closed for short) if L^+ is closed w.r.t. **Sb** and L^- is closed w.r.t. **RS**. Clearly, for each s-closed pair $\text{L} = (\text{L}^+, \text{L}^-)$, there is a unified deductive system $\mathcal{D}$ such that $\text{L}_{\mathcal{D}}^+ = \text{L}^+$ and $\text{L}_{\mathcal{D}}^- = \text{L}^-$, and we say that $\mathcal{D}$ is *L-adequate* for L .

For instance, unified deductive system $\mathcal{C}$ from Example 3 is L -adequate for (L^+, L^-) , where L^+ is a set of all classical tautologies, and L^- is a set of all non-tautologies.

Example 4. Let IPC^+ be a set of statements obtained by preceding by $\oplus$ every axiom of the Intuitionistic propositional logic. Let us consider the unified deductive system $\mathcal{D} = \langle \text{Ax}; \text{R}^s \rangle$, where $\text{Ax}^+ = \text{IPC}^+ \cup \{\oplus((\neg q \rightarrow p) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p))\}$, $\text{Ax}^- = \{\ominus(\neg \neg p \rightarrow p)\}$ and $\text{R}^s = \{\text{MP, MT, Sb, RS}\}$. The unified logic defined by $\mathcal{D}$ gives an L -complete axiomatization of the logic of the three element Heyting algebra.

4.2. Derivation of rules

Let R be a set of rules and $r = \Gamma/\Delta$ be a rule. We say that r is *derivable* from R or r is *reducible* to R (in symbols $\text{R} \vdash r$), if there is a $\langle \Gamma; \text{R} \rangle$ -prot inference all extendable leaves of which contain statements only from Δ , that is, if $\Gamma \vdash_{\text{R}} \Delta$.

If R is a set of rules and r, r' are the rules, we say that r is *derivable from* r' or r is *reducible* to r' *relative* to R (in symbols $r' \vdash_{\text{R}} r$), if

$$\text{R}, r' \vdash r.$$

$r' \vdash_{\text{R}} r$ means that in any prot inference, every application of rule r can be replaced with the suitable applications of rules R and r' . In other words, rule r can be eliminated from any prot inference and replaced by rules R, r' .

The rules **Co** and **Fu** allow to derive the different variations of a given rule from each other. Let $\text{S} = \{\text{Co, Fu}\}$.

Proposition 8. The following holds:

- (a) $Sb \vdash_S RS$;
- (b) $MP \vdash_S MT$;

If the signature contains $\neg$ and $Ex = \oplus p, \oplus \neg p / \blacktriangle$ and $S' = S \cup \{Sb, Ex\}$, then

- (c) $\blacktriangledown \vdash_{S'} \ominus p$.

We need to prove (a) $\ominus \sigma A \vdash_S \ominus A$, (b) $\oplus(A \rightarrow B), \ominus B \vdash_S \ominus A$, and (c) $\blacktriangledown \vdash_{S'} \ominus p$. The proofs are depicted in Table 1.

Table 1. Reductions

Example 5. Łukasiewicz's classical logic with refutation can be defined by any of the following unified deductive systems:

- axioms: $\{\oplus p \rightarrow (q \rightarrow p),$
 $\oplus(p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))),$
 $\oplus((\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p))\}$
- anti-axiom: $\{\ominus p\}$
- rules: $\{MP, Co, Fu, Sb\}$

or

- axioms: $\{\oplus p \rightarrow (q \rightarrow p),$
 $\oplus(p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))),$
 $\oplus((\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p))\}$
- rules: $\{MP, Co, Fu, Sb, Ex\}$

Remark 2. There is a very important difference between the traditional deductive systems in which axioms are formulas and rules allow to derive formulas from formulas, and unified deductive systems. In the traditional deductive system, any axiom A can be converted into a rule $\emptyset/A$. In the unified deductive systems, any positive axiom $\oplus A$ can be converted into a rule $\nabla/\oplus A$, while anti-axioms may not be replaceable by the rules. For instance, $\ominus p$ cannot be replaced either by $\nabla/\ominus p$, or by $\ominus p/\blacktriangle$.

4.3. Standard deductive systems

A unified deductive system $\mathcal{D}$ is *conventional* if it defines a conventional unified logic, that is, full and coherent logic. $\mathcal{D}$ is conventional if and only if the rules **Co** and **Fu** are admissible in it.

Definition 6. Let $\mathcal{D} = \langle \text{Ax}; \text{R}^s \rangle$ be a unified deductive system. A rule r is *derivable from a set of rules R' relative to $\mathcal{D}$* (in symbols, $\text{R}' \vdash_{\mathcal{D}} r$), if $\text{R}' \vdash_{\text{R}} r$. A rule r is derivable in $\mathcal{D}$ if $\vdash_{\mathcal{D}} r$.

Let us point out that to derive rule r from R' relative to $\mathcal{D}$, we use only m-rules of $\mathcal{D}$ and we do not use axioms or rules **Sb**, **RS**. The reason for that is that use of anti-axioms in the derivations of rules may lead to undesirable consequences. For instance, the use of anti-axiom $\ominus p$ in the first unified deductive system from Example 5 would allow to derive the rule $\nabla/\ominus p$ and subsequently, to derive $\ominus(p \rightarrow p)$.

Definition 7. A unified deductive system is *standard* if the rules **Co** and **Fu** are derivable in it. In particular, every unified deductive system in which the rules **Co** and **Fu** are postulated is standard.

In contrast to conclusive rules, a derivable inconclusive rule needs not to be admissible. The rule of fullness $\text{Fu} = \nabla/\oplus p, \ominus p$ is inconclusive, thus, even if it is derivable in a unified logic, it may be not admissible for it. On the other hand, because rule $\text{Co} = \oplus p, \ominus p/\blacktriangle$ is conclusive, it is admissible in every standard deductive system and hence, every standard deductive system is coherent (while needs not to be full.)

Theorem 1 (About Symmetry). *Let $\text{S} = \{\text{Co}, \text{Fu}\}$, Γ, Δ be finite sets of statements and α be a statement. Then,*

- (a) *if $\alpha, \Gamma \vdash_{\text{S}} \Delta$, then $\Gamma \vdash_{\text{S}} \Delta, \bar{\alpha}$;*
- (b) *if $\Gamma \vdash_{\text{S}} \Delta, \bar{\alpha}$, then $\alpha, \Gamma \vdash_{\text{S}} \Delta$.*

Proof. (a) If $\alpha = \nabla$, then by Propositions 6(a) and (b), $\nabla, \Gamma \vdash_{\text{S}} \Delta$ if and only if $\Gamma \vdash_{\text{S}} \Delta$ and if and only if $\Gamma \vdash_{\text{S}} \Delta, \blacktriangle$.

If $\alpha = \blacktriangle$, by Proposition 6(d) and (c), $\blacktriangle, \Gamma \vdash_S \Delta$ and $\Gamma \vdash_S \blacktriangledown, \Delta$ both hold.

Suppose that α is a proper statement and $\mathcal{J}$ is a $\langle \alpha, \Gamma; S \rangle$ -protoinference of Δ . Without loosing generality, one can assume that the root of $\mathcal{J}$ marked by $\blacktriangledown$ has a unique successor marked by α . The proof follows from the observation that one can construct a $\langle \Gamma; S \rangle$ -protoinference $\mathcal{J}'$ of $\bar{\alpha}, \Delta$ by applying Fu to $\blacktriangledown$ and then, attaching $\mathcal{J}$ by identifying $\oplus \alpha$ with the node of $\mathcal{J}$ marked by α :

It is not hard to see that the obtained tree $\mathcal{J}'$ is indeed a $\langle \Gamma; S \rangle$ -protoinference of $\bar{\alpha}, \Delta$.

(b) If $\alpha = \blacktriangledown$, by Proposition 6(c) and (d), $\Gamma \vdash_S \blacktriangledown, \Delta$ and $\blacktriangle, \Gamma \vdash_S \Delta$ both hold.

If $\alpha = \blacktriangle$, then by Propositions 6(b) and (a), $\Gamma \vdash_S \Delta, \blacktriangle$ if and only if $\Gamma \vdash_S \Delta$ if and only if $\blacktriangledown, \Gamma \vdash_S \Delta$.

Suppose that α is a proper statement and $\mathcal{J}$ is a $\langle \Gamma; S \rangle$ -protoinference of $\Delta, \bar{\alpha}$. Consider two cases: (i) $\mathcal{J}$ contains a leaf labeled by $\bar{\alpha}$, and (ii) $\mathcal{J}$ does not contain a leaf labeled by $\bar{\alpha}$.

(i) The proof follows from the following observation: to construct a $\langle \alpha, \Gamma; S \rangle$ -protoinference $\mathcal{J}'$ of Δ , because α is an assumption, one can extend every leaf of $\mathcal{J}$ labeled by $\bar{\alpha}$ with a leaf labeled by α and then apply Co:

It is not hard to see that the obtained tree $\mathfrak{J}'$ is indeed a $\langle \alpha, \Gamma; \mathbf{S} \rangle$ -protoinference of Δ .

(ii) Suppose that $\mathfrak{J}$ does not contain a leaf labeled by $\bar{\alpha}$. Thus, only members of Δ appear in the leaves of $\mathfrak{J}$ and hence, $\mathfrak{J}$ is a $\mathcal{D}$ -protoinference of Δ from Γ . Then, by (1), $\mathfrak{J}$ is a $\mathcal{D}$ -protoinference of Δ from α, Γ . ■

Theorem 1 yields some important consequences.

Corollary 2. Let $\mathcal{D}$ be a standard unified deductive system, Γ, Δ be finite nonempty sets of statements and α be a statement. Then,

- (a) if $\alpha, \Gamma \vdash_{\mathcal{D}} \Delta$, then $\Gamma \vdash_{\mathcal{D}} \Delta, \bar{\alpha}$;
- (b) if $\Gamma \vdash_{\mathcal{D}} \Delta, \bar{\alpha}$, then $\alpha, \Gamma \vdash_{\mathcal{D}} \Delta$.

Corollary 3. Let $\mathcal{D}$ be a standard unified deductive system and Γ, Δ be finite sets of statements. Then,

$$\text{if } \Gamma \vdash_{\mathcal{D}} \Delta, \text{ then } \bar{\Delta} \vdash_{\mathcal{D}} \bar{\Gamma}.$$

In particular, for any statements α and β ,

$$\text{if } \alpha \vdash_{\mathcal{D}} \beta, \text{ then } \bar{\beta} \vdash_{\mathcal{D}} \bar{\alpha}.$$

Corollary 4. Let $\mathcal{D}$ be a standard unified deductive system, Γ, Δ be finite nonempty sets of statements and α be a statement. Then,

$$\Gamma, \alpha / \Delta \vdash_{\mathcal{D}} \Gamma / \Delta, \bar{\alpha} \quad \text{and} \quad \Gamma / \Delta, \alpha \vdash_{\mathcal{D}} \Gamma, \bar{\alpha} / \Delta.$$

Two deductive systems $\mathcal{D}_0$ and $\mathcal{D}_1$ are *equivalent* if for any finite sets of statements Γ and Δ ,

$$\Gamma \vdash_{\mathcal{D}_0} \Delta \text{ if and only if } \Gamma \vdash_{\mathcal{D}_1} \Delta.$$

Corollary 5. Every standard unified deductive system has an equivalent standard unified deductive system all the rules of which, except for **Co** and **Fu**, have no negative statements.

Let us note that Corollary 5 states that any unified deductive system can be transformed into a deductive system containing (except for **Fu** and **Co**) only the ordinary rules (without negative statements) and in a way, such a deductive system realizes the Carnap approach to proving rejections with **Co** used instead of Modus Tollens. If all rules of a unified deductive system except for **Fu** and **Co** are positive, **Fu** can be eliminated from any inference, while, because **Co** has $\blacktriangle$ as a single conclusion, every applications of **Co** is always the last application of a rule on a branch of the inference.

Corollary 6. Every standard unified deductive system has an equivalent standard unified deductive system, every m-rule of which, except maybe for Fu , is conclusive.

Recall that all derivable conclusive rules are admissible. Thus, Corollary 7 entails the following.

Corollary 7. In every standard unified deductive system, all postulated m-rules except maybe for Fu , are admissible.

4.4. Conclusion

As we saw, the rules of fullness and coherence are playing a special role in unified deductive system. Despite the fact that they are the reverse of each other, these rules express different principles, namely,

- the rule of fullness $\nabla / \oplus p, \ominus p$ (and not $\nabla / \oplus p, \oplus \neg p$, or $\nabla / \oplus (p \vee \neg p)$) expresses the Law of Excluded Middle:

Every proposition is asserted or rejected;

- accordingly, the rule of coherence $\oplus p, \ominus p / \blacktriangle$ expresses the Law of Non-Contradiction:

No proposition is asserted and rejected at the same time.

The laws of Excluded Middle and Non-Contradictions are not about negation: we may have them for the systems without negation.

Postulating rules of fullness and coherence in a deductive system makes derivations in such a system “reversible”, and this allows to reduce any rule (except of the rule of coherence) to rules without negative assumptions or conclusions. In addition, any rule in such a system (except for the rules of fullness and coherence) can be reduced to a conclusive rule. This explains perhaps why we do not use the multiple-conclusion rules that often: implicitly, outside of the formal deductive system, we use the rules of fullness and coherence.

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