BRUNO R. MENDONÇA, WALTER A. CARNIELLI

New semantics for urn logics: taming the enduring scandal of deduction

Bruno R. Mendonça

University of Campinas,

R. Sérgio Buarque de Holanda, 251 - Cidade Universitária, Campinas - SP, 13083-859, Brazil.

E-mail: bruno.amos.mendonca@gmail.com

Walter A. Carnielli

University of Campinas,

R. Sérgio Buarque de Holanda, 251 - Cidade Universitária, Campinas - SP, 13083-859, Brazil.

E-mail: walter.carnielli@cle.unicamp.br

Abstract: The traditional theory of semantic information, originally proposed by [Bar-Hillel, Carnap, 1953], provides a versatile and pretty plausible conception of what kind of thing semantic information is. It embodies, however, the so-called “scandal of deduction”, a thesis according to which logical truths are informationally empty. The scandal of deduction is problematic because it contradicts the fact that ordinary reasoners often do not know whether or not a given sentence is a logical truth. Hence, it is plausible to say, at least from the epistemological standpoint of those reasoners, that such logical sentences are really informative. In order to improve over traditional theory, we can replace its classical metatheory by the so-called *urn logics*, non-standard systems of logic (described in detail below) that better describe the epistemological standpoint of ordinary reasoners. Notwithstanding, the application of such systems to the problem of semantic information faces some challenges: first, we must define truth-conditional semantics for these systems. Secondly, we need to precisely distinguish two systems of urn logic, namely, perfect and imperfect urn logics. Finally, we need to prove characterization theorems for both systems of urn logic. In this paper we offer original (and hopefully, elegant) solutions for all such problems.

Keywords: semantic information, scandal of deduction, logical knowledge, urn logics, truth-conditional semantics, characterization theorems, Hintikka normal forms, non-classical logics

For citation: Mendonça R., Carnielli A. “New semantics for urn logics: taming the enduring scandal of deduction”, *Logicheskie Issledovaniya* / *Logical Investigations*, 2020, Vol. 26, No. 1, pp. 91–109. DOI: 10.21146/2074-1472-2020-26-1-91-109

Dedicated to the memory of Professor Alexander Karpenko

1. Introduction

There is a wide debate in the current logical literature on the nature of semantic information and how it can be measured. Traditional theory of semantic information, originally proposed by [Bar-Hillel, Carnap, 1953], equates the semantic information of a sentence with the number of different models that falsify it. A problematic consequence of this approach is the so-called “scandal of deduction” [Hintikka, 1970a]: according to this thesis, logical truths are informationally empty, due to the absence of falsifying models.

The scandal of deduction seems to be in conflict with the recognizable fact that reasoners often do not know whether or not a given sentence is a logical truth. For instance, beginning students of set theory may find trouble to recognize that Russell’s paradox — a problematic consequence of naive set theory — involves a contradiction. In more precise terms, these individuals ignore that the sentence $\neg\exists x\forall y(y \in x \leftrightarrow y \notin y)$ is logically valid. So, it is plausible to assume that, since there are logical truths whose validity is *prima facie* ignored by ordinary reasoners, at least from the epistemological standpoint of these individuals, such logical sentences are not uninformative.

Mainstream attempts to solve the scandal of deduction generally propose replacing traditional theory by more or less distorted alternatives (as, for instance, in [Floridi, 2004]). However, some of these proposals seem to throw the baby out with the bathwater. Traditional theory offers a flexible and persuasive conception of the kind of information associated with the meaning of sentences, and should be preserved. Fortunately, it is possible to simultaneously save traditional theory and block the scandal of deduction by revising the logical metatheory on which our theory of semantic information is based on.

As mentioned above, the scandal of deduction contradicts the fact that at least some logical truths are really informative when considered from the epistemological standpoint of ordinary reasoners. So, we could block this thesis by adopting some logical theory that better describes the semantic knowledge (i.e., the knowledge of truth-conditions, by assuming a truth-conditional approach to semantics) associated with the semantic competence of ordinary reasoners. This new logical theory should be able to support the idea that, for some logical truths, an ordinary reasoner does not know its truth-conditions; more specifically, it should point out that the epistemological stance of an ordinary reasoner cannot exclude some impossible models that falsify the logical truth in question.

Urn logics, a generalization of first-order logic introduced in [Rantala, 1975], are good candidates to fulfill such an explanatory role. Drawing balls from an

urn with or without replacement is a classical problem in probability and statistics. An urn problem is a mental experiment in which colored balls that represent events are withdrawn and whose probabilities should be determined. An urn model is a probability distribution, or a family of probability distributions associated with urn problems.

Now, the intuition behind urn logic is to think of the domain of quantification in a way analogous to urns of balls used in probability theory. The metaphor here is to think of quantifying over objects in a domain as being analogous to drawing balls from an urn. In the standard models of first-order logic the domain of quantification stays fixed as we ‘draw’ elements from the domain. But we can think of models where the domain of quantification changes in the course of evaluating a formula, as probability urns where balls can be replaced, or new balls can be added.

Basically, the fragments of classical logic introduced by this new procedure lose the capacity to express dependence relations between nested quantifiers in a given formula.¹ Consequently, these systems can formalize the epistemological standpoint of someone who does not know that some first-order valid sentence ϕ is a logical truth because this individual is not aware of the dependence relations holding between nested quantifiers occurring in ϕ .

The application of urn logics to the problem of semantic information, however, faces some technical challenges. First, whereas traditional theory presupposes truth-conditional semantics, urn logics were originally defined in game-theoretic terms. Secondly, there are at least two different systems of urn logic which generate different versions of the traditional theory of semantic information, but the literature does not precisely differentiate such systems (in this paper these systems are called *perfect* and *imperfect* urn logics, for reasons that will become clear soon). Finally, it is not clear exactly which logical truths are semantically informative in the context of urn logics. In order to make this point clear, we need to characterize the set of logical truths of urn logics.

To solve these challenges, we define in this paper a truth-conditional semantics for urn logics that is able to precisely separate their perfect and imperfect versions (section 2). Further, we present a characterization theorem for the set of validities of perfect urn logic, and draw some remarks on how it is possible to obtain a corresponding result for imperfect urn logic (section 4). Such a theorem relies on an important auxiliary result to be presented in section 3, namely, the existence of Hintikka normal forms for urn logics. All of these

¹Urn semantics should not be confused with *IF-logic* [Tulenheimo, 2018] and related systems. Whilst IF-logic extends first order logic with the capacity of expressing independence / dependence relations between nested quantifiers, urn semantics deals with a fragment of classical logic.

results are original contributions to the study of urn logics — in particular, the characterization theorems introduced in section 4 provide a full generalization of a partial result presented in [Rantala, 1975, pp. 470–472].

2. Truth-conditional semantics for urn logics

Generally speaking, the idea behind urn logics is that two nested quantifiers Q and K occurring in some formula may have different quantificational domains and there are not enough logical resources in these systems to express dependence relations between them. Thus, for instance, in urn logics the quantifiers in $\forall x \exists y (x < y)$ may have different domains and there is no way to force dependence relations between them.

We can formalize this idea in the following way. In what follows, let M be a non-empty set. For any elements $a_0, \dots, a_n$ of M , let $B(a_0, \dots, a_n)$ be some non-empty subset of M . We call such sets $B(a_0, \dots, a_n)$ *choice sets* and there is no specific restrictions on how they are supposed to be concretely defined.

For every $n \in \mathbb{N}$, a *set of choice n -sequences* $\mathfrak{M}_n$ of M is such that the following holds:

1. $\mathfrak{M}_0$ is a set of unary sequences of M ;
2. For each $\langle a_0, \dots, a_n \rangle \in \mathfrak{M}_n$, consider a choice set $B(a_0, \dots, a_n)$ of M . Then, $\mathfrak{M}_{n+1} = \{ \langle a_0, \dots, a_n, b \rangle : \langle a_0, \dots, a_n \rangle \in \mathfrak{M}_n, \text{ for every } b \in B(a_0, \dots, a_n) \}$.

Further, let $\mathfrak{M} = \bigcup_{n \in \mathbb{N}} \mathfrak{M}_n$ be the *set of choice sequences* of M . In order to better visualize the meaning of these concepts, consider the following example: assume M is the set of natural numbers and let $\mathfrak{M}_0 = \{1\}$ and, for any sequence of natural numbers $a_0, \dots, a_n$, $B(a_0, \dots, a_n) = \{2a_n\}$. Then $\mathfrak{M}$ collects the initial segments of the geometric progression starting with 1 and with common ratio 2.

Based on those notions, [Rantala, 1975] defines a system of urn logic game-theoretically.

Definition 1. Let M be a classic structure, $\mathfrak{M}$ a set of choice sequences of M and ϕ a formula of the language of M . A urn game semantics $G(M, \mathfrak{M}, \phi)$ is similar to classic game semantics except in the following clauses. Assume that ϕ' is the subformula of ϕ considered in the i -th round of a match p of $G(M, \mathfrak{M}, \phi)$ and the choice k -sequence $\langle a_0, \dots, a_k \rangle$ has been chosen in the previous rounds:

1. If ϕ' is $\exists x \psi$, then the player who holds ϕ' in the i -th round of p holds $\psi(b/x)$ in the $i + 1$ -th round of p , for some b of his own choice and such that $\langle a_0, \dots, a_k, b \rangle \in \mathfrak{M}_{k+1}$;

2. If ϕ' is $\forall x\psi$, then the player who holds ϕ' in the i -th round of p holds $\psi(b/x)$ in the $i + 1$ -th round of p , for some b of other player's choice and such that $\langle a_0, \dots, a_k, b \rangle \in \mathfrak{M}_{k+1}$;

The difference between classic and urn game semantics lies essentially in the behavior of quantifiers: whilst in the classic game a player who chooses witnesses in the model might choose them looking in the entire domain, in urn game semantics the player must limit herself to choice sequences only.

Definition 1 describes a game of perfect information, i.e., in order to choose witnesses in the model, the player must verify which witnesses were chosen in the previous rounds of the match. We can define a variant game of imperfect information by adding the condition that, for any $n \in \mathbb{N}$, for any two choice n -sequences $\bar{a}$ and $\bar{b}$, $B(\bar{a}) = B(\bar{b})$. In this case, the player does not need anymore to consider the particular sequence of witnesses that have appeared previously in the game because any such sequence determines the same choice set. To avoid confusion, we call the logical systems which are defined by these two different urn game semantics perfect and imperfect urn logics, respectively.

The greatest obstacle for the construction of a truth-conditional semantics for urn logics is the non-compositional character of these systems [Cresswell, 1982, pp. 128–129]. We can solve this issue by relativizing the satisfaction of formulas in the following way.

Definition 2. Let M be a classic structure with a set of choice sequences $\mathfrak{M}$. For every $n \in \mathbb{N}$ and for any formula ϕ of the language of M , the pair $(M, \mathfrak{M})$ n -satisfies ϕ with respect to $\langle a_0, \dots, a_{n-1} \rangle$, in symbols $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi$, if the following holds:

- If ϕ is atomic formula, $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi \Leftrightarrow M$ classically satisfies ϕ ;
- If ϕ is $\psi \wedge \gamma$, $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi$ if and only if $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \psi$ and $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \gamma$ (an analogous clause holds for disjunction);
- If ϕ is $\neg\psi$, $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi \Leftrightarrow M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \not\models \psi$;
- If ϕ is $\exists x\psi$, $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi \Leftrightarrow M_{\mathfrak{M}}, a_0, \dots, a_{n-1}, b \models \psi(b/x)$, for some b such that $\langle a_0, \dots, a_{n-1}, b \rangle \in \mathfrak{M}_n$.
- If ϕ is $\forall x\psi$, $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \phi \Leftrightarrow M_{\mathfrak{M}}, a_0, \dots, a_{n-1}, b \models \psi(b/x)$, for every b such that $\langle a_0, \dots, a_{n-1}, b \rangle \in \mathfrak{M}_n$.²

²Strictly speaking, we do not need to present a specific definition of satisfaction for universal formulas since $\forall$ is definable in urn logics as $\neg \exists \neg$. But, since in the following we will extensively consider universal formulas, this specific clause shows itself to be important.

Finally, $(M, \mathfrak{M})$ satisfies ϕ , in symbols $M_{\mathfrak{M}} \models \phi$, if and only if $(M, \mathfrak{M})$ 0-satisfies ϕ with respect to $\emptyset$.

In the next theorem, following the usual conventions of the literature on game theory, we refer to the players of an urn game semantics as *Abelard* and *Eloise*. Eloise is the player that initially holds the formula ϕ in a match of the game $G(M, \mathfrak{M}, \phi)$.

Theorem 1. *Let M be a classic structure with a set of choice sequences $\mathfrak{M}$, and let ϕ be a formula of the language of M . The following are equivalent:*

1. $M_{\mathfrak{M}} \models \phi$;
2. *Eloise has a winning strategy in $G(M, \mathfrak{M}, \phi)$.*

Proof. $(1 \Rightarrow 2)$ Assume 1. We can define a winning strategy σ for Eloise in $G(M, \mathfrak{M}, \phi)$ as follows. Suppose that, for some $n \in \mathbb{N}$, we have defined σ in the first i -th rounds of a match p of $G(M, \mathfrak{M}, \phi)$. Moreover, consider that the choice k -sequence $\langle a_0, \dots, a_k \rangle$ has been chosen in those rounds of p :

- Assume Abelard holds $\psi \wedge \gamma$ in the i -th round of p . If $M_{\mathfrak{M}, a_0, \dots, a_k} \not\models \psi$, then Eloise demands Abelard to hold ψ in the $i + 1$ -th round of p ; otherwise, Eloise demands Abelard to hold γ ;
- Assume Eloise holds $\psi \vee \gamma$ in the i -th round of p . If $M_{\mathfrak{M}, a_0, \dots, a_k} \models \psi$, then Eloise demands herself to hold ψ in the $i + 1$ -th round of p ; otherwise, Eloise demands herself to hold γ ;
- Assume Abelard holds $\forall x\psi$ in the i -th round of p . If there is some b such that $M_{\mathfrak{M}, a_0, \dots, a_k, b} \not\models \psi(b/x)$ such that $\langle a_0, \dots, a_k, b \rangle \in \mathfrak{M}_{k+1}$, then Eloise demands Abelard to hold $\psi(b/x)$; otherwise, Eloise demands Abelard to hold $\psi(c/x)$, for some arbitrary c such that $\langle a_0, \dots, a_k, c \rangle \in \mathfrak{M}_{k+1}$;
- Assume Eloise holds $\exists x\psi$ in the i -th round of p . If there is some b such that $M_{\mathfrak{M}, a_0, \dots, a_k, b} \models \psi(b/x)$ such that $\langle a_0, \dots, a_k, b \rangle \in \mathfrak{M}_{k+1}$, then Eloise demands herself to hold $\psi(b/x)$; otherwise, Eloise demands herself to hold $\psi(c/x)$, for some arbitrary c such that $\langle a_0, \dots, a_k, c \rangle \in \mathfrak{M}_{k+1}$.

By induction on the indexes of the rounds of any match p in which Eloise follows strategy σ , we can verify that p is a winning case for her. Consequently, σ is a winning strategy for Eloise.

$(2 \Rightarrow 1)$ Assuming $M_{\mathfrak{M}} \not\models \phi$, we can define a winning strategy σ' for Abelard considering a set of directives dual to those defining σ . By a similar argument it is possible to prove that Eloise loses any match in which Abelard follows σ' . Consequently, Eloise has no winning strategy in $G(M, \mathfrak{M}, \phi)$. ■

3. Hintikka normal forms for urn logics

The existence of Hintikka normal forms is a fundamental property of classical logic with a variety of applications: Hintikka normal forms characterize tableau-like deductive systems [Hintikka, 1965], generate truth tables for quantificational logic [Freire, 2015] and provide an essential tool for the construction of theories of semantic information [Hintikka, 1970b]. Particularly in the case of urn logics, it is possible to obtain characterization theorems for these systems based on the syntactic structure of special kinds of Hintikka normal forms, as we will show in section 4 below.

Let an *unnested formula* be a formula whose terms have complexity at most 1. For a finite set of formulas $\Psi = \{\psi_0, \dots, \psi_n\}$, let $\bigwedge \Psi$ denote $\bigwedge_{i \leq n} \psi_i$ and $\bigvee \Psi$ denote $\bigvee_{i \leq n} \psi_i$. Let $\neg \Psi = \{\neg \psi : \psi \in \Psi\}$ and $\exists x \Psi = \{\exists x \psi : \psi \in \Psi\}$. Moreover, let the *quantifier rank* of ϕ , $qr(\phi)$, be as follows:

- For every atomic formula ϕ , $qr(\phi) = 0$;
- If ϕ is $\neg \psi$, then $qr(\phi) = qr(\psi)$;
- If ϕ is either $\psi \wedge \gamma$ or $\psi \vee \gamma$, then $qr(\phi) = \max \{qr(\psi), qr(\gamma)\}$;
- If ϕ is either $\forall x \psi$ or $\exists x \psi$, then $qr(\phi) = qr(\psi) + 1$.

Finally, let $\Phi(x_0, \dots, x_n)$ be the set of unnested atomic formulas of the considered language with variables within $x_0, \dots, x_n$.

Definition 3. Let $\mathcal{L}$ be a finite first order language. For every $n, m \in \mathbb{N}$, a state description $\theta_m(x_0, \dots, x_n)$ of $\mathcal{L}$ is such that:

- $\theta_0(x_0, \dots, x_n)$ is

$$\bigwedge \Sigma \wedge \bigwedge \neg[\Phi(x_0, \dots, x_n) - \Sigma],$$

for some $\Sigma \subseteq \Phi(x_0, \dots, x_n)$;

- Now, assume that $m > 0$ and the set Θ of state descriptions of $\mathcal{L}$ with quantifier rank $m - 1$ and free-variables $x_0, \dots, x_n, y_m$ has been defined: the state description $\theta_m(x_0, \dots, x_n)$ is $(\bigwedge \exists y_m \Gamma) \wedge (\forall y_m \bigvee \Gamma)$, for some set $\Gamma \subseteq \Theta$.

For any formula ϕ of $\mathcal{L}$ with quantifier rank m and free-variables $x_0, \dots, x_n$, ϕ is a Hintikka normal form if and only if ϕ is $\bigvee \Gamma$, for some set Γ of state descriptions with quantifier rank m and free-variables $x_0, \dots, x_n$.

In classical logic, we can prove the existence of Hintikka normal forms by showing that satisfiable state descriptions define *partial isomorphisms* [Hodges, 1997, pp. 84–85]. We can generalize this strategy for the case of urn logics.

Definition 4. Let M and N be classic structures with sets of choice sequences $\mathfrak{M}$ and $\mathfrak{N}$, respectively. There is a partial isomorphism $M_{\mathfrak{M}} \simeq N_{\mathfrak{N}}$ if and only if there is a poset H of isomorphisms between finite parts of M and N such that the following holds:

1. The first element ι_0 in H is $\emptyset$;
2. Assume that, for some $n \in \mathbb{N}$, there is an isomorphism ι_n between substructures $\{a_0, \dots, a_{n-1}\} \subseteq M$ and $\{b_0, \dots, b_{n-1}\} \subseteq N$ such that $\iota_n(a_i) = b_i$, $\langle a_0, \dots, a_{n-1} \rangle \in \mathfrak{M}$ and $\langle b_0, \dots, b_{n-1} \rangle \in \mathfrak{N}$.
 - For any $a \in M$ such that $\langle a_0, \dots, a_{n-1}, a \rangle \in \mathfrak{M}$, there is $b \in N$ such that $\langle b_0, \dots, b_{n-1}, b \rangle \in \mathfrak{N}$ and there is an isomorphism ι_{n+1} between $\{a_0, \dots, a_{n-1}, a\}$ and $\{b_0, \dots, b_{n-1}, b\}$ such that $\iota_n \subseteq \iota_{n+1} \in H$;
 - For any $b \in N$ such that $\langle b_0, \dots, b_{n-1}, b \rangle \in \mathfrak{N}$, there is $a \in M$ such that $\langle a_0, \dots, a_{n-1}, a \rangle \in \mathfrak{M}$ and there is an isomorphism ι_{n+1} between $\{a_0, \dots, a_{n-1}, a\}$ and $\{b_0, \dots, b_{n-1}, b\}$ such that $\iota_n \subseteq \iota_{n+1} \in H$;
3. H is the smallest poset satisfying conditions 1 and 2.

Moreover, we say that there is a k -partial isomorphism $M_{\mathfrak{M}} \simeq_k N_{\mathfrak{N}}$ if and only if condition 2 holds at least for every $n < k$.

Lemma 1. Let M be a classic structure of a finite language and let $\mathfrak{M}$ be a set of choice sequences of M . Then, for every $m, n, k \in \mathbb{N}$, for any elements $a_0, \dots, a_n \in M$, there is a unique state description $\theta_m(x_0, \dots, x_n)$ such that $M_{\mathfrak{M}} \models \theta_m(a_0, \dots, a_n)$.

Proof. Proof by induction on $m \in \mathbb{N}$. In the base case, $\theta_0(x_0, \dots, x_n)$ is

$$\bigwedge \Sigma \wedge \bigwedge \neg[\Phi(x_0, \dots, x_n) - \Sigma],$$

for $\Sigma = \{\phi \in \Phi(x_0, \dots, x_n) : M_{\mathfrak{M}} \models \phi(a_0, \dots, a_n)\}$. In the inductive step, consider $\Gamma = \{\theta_m(x_0, \dots, x_n, y) : M_{\mathfrak{M}} \models \theta_m(a_0, \dots, a_n, b), \text{ for some } b \text{ such that } \langle a_0, \dots, a_n, b \rangle \in \mathfrak{M}\}$. Then, $\bigwedge \exists y \Gamma \wedge \forall y \bigvee \Gamma$ is the relevant state description. ■

Lemma 2. Let $\mathcal{L}$ be a finite language, M and N be classic structures of $\mathcal{L}$. Consider $\mathfrak{M}$ and $\mathfrak{N}$ sets of choice sequences of M and N , respectively. For any $k \in \mathbb{N}$, the following are equivalent:

1. $M_{\mathfrak{M}} \simeq_k N_{\mathfrak{N}}$;
2. For every $m \leq k$, there is a unique state description θ_m of $\mathcal{L}$ that is satisfied by $(M, \mathfrak{M})$ and $(N, \mathfrak{N})$.

Proof. (1 $\Rightarrow$ 2) Assume 1. Item 2 is a consequence of the following fact:

- (*) For any m and n such that $m + n \leq k$, for any state description $\theta_m(x_0, \dots, x_{n-1})$ and for any $\langle a_0, \dots, a_{n-1} \rangle \in \mathfrak{M}$,

$$M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \theta_m(a_0, \dots, a_{n-1}) \Leftrightarrow$$

$$N_{\mathfrak{N}}, \iota_n(a_0), \dots, \iota_n(a_{n-1}) \models \theta_m(\iota_n(a_0), \dots, \iota_n(a_{n-1}))$$

for ι_n in the poset H of isomorphisms that defines $M_{\mathfrak{M}} \simeq_k N_{\mathfrak{N}}$.

The proof of (*) follows by induction on m . Assume that the property holds for some $m < k$. For any n such that $m + n < k$, consider a state description $\theta_m(x_0, \dots, x_{n-1}, y)$. We can show that $(M, \mathfrak{M})$ and $(N, \mathfrak{N})$ agree in all formulas of the form $\exists y \theta_m(x_0, \dots, x_{n-1}, y)$ in the following way.

Suppose that $M_{\mathfrak{M}}, a_0, \dots, a_{n-1} \models \exists y \theta_m(a_0, \dots, a_{n-1}, y)$ (the proof in the other direction works in exactly the same way). Therefore,

$$M_{\mathfrak{M}}, a_0, \dots, a_{n-1}, a \models \theta_m(a_0, \dots, a_{n-1}, a)$$

for some a such that $\langle a_0, \dots, a_{n-1}, a \rangle \in \mathfrak{M}$.

By inductive hypothesis and 1, there is $\iota_{n+1} \in H$ such that

$$N_{\mathfrak{N}}, \iota_{n+1}(a_0), \dots, \iota_{n+1}(a_{n-1}), \iota_{n+1}(a) \models \theta_m(\iota_{n+1}(a_0), \dots, \iota_{n+1}(a_{n-1}), \iota_{n+1}(a)).$$

So, $N_{\mathfrak{N}}, \iota_{n+1}(a_0), \dots, \iota_{n+1}(a_{n-1}) \models \exists y \theta_m(\iota_{n+1}(a_0), \dots, \iota_{n+1}(a_{n-1}), y)$.

Consider the set

$$\Gamma = \{\theta_m(x_0, \dots, x_{n-1}, y) : M_{\mathfrak{M}}, a_0, \dots, a_{n-1}, a \models \theta_m(a_0, \dots, a_{n-1}, a),$$

for some a such that $\langle a_0, \dots, a_{n-1}, a \rangle \in \mathfrak{M}\}$.

$(M, \mathfrak{M})$ and $(N, \mathfrak{N})$ both satisfy $(\bigwedge \exists y \Gamma) \wedge (\forall y \bigvee \Gamma)$. Moreover, by Lemma 6 this is the only state description that we must consider. This completes the proof of (*), 2 being a subcase of it.

(2 $\Rightarrow$ 1) Assume 2. We can construct $M_{\mathfrak{M}} \simeq_k N_{\mathfrak{N}}$ in the following way. For some $m < k$, suppose $M_{\mathfrak{M}} \simeq_m N_{\mathfrak{N}}$ has been defined. Let ι_m be one of the isomorphisms which define $M_{\mathfrak{M}} \simeq_m N_{\mathfrak{N}}$ and let the domain of ι_m be $\{a_0, \dots, a_{m-1}\}$ such that $\langle a_0, \dots, a_{m-1} \rangle \in \mathfrak{M}$.

Without loss of generality, fix some element a such that $\langle a_0, \dots, a_{m-1}, a \rangle \in \mathfrak{M}$ and consider the state description $\theta_{k-(m+1)}(x_0, \dots, x_{m-1}, y)$ such that

$$M_{\mathfrak{M}}, a_0, \dots, a_{m-1}, a \models \theta_{k-(m+1)}(a_0, \dots, a_{m-1}, a).$$

By 2 and the definition of $M_{\mathfrak{M}} \simeq_m N_{\mathfrak{N}}$,

$$N_{\mathfrak{N}}, \iota_m(a_0), \dots, \iota_m(a_{m-1}), b \models \theta_{k-(m+1)}(\iota_m(a_0), \dots, \iota_m(a_{m-1}), b),$$

for some b such that $\langle \iota_m(a_0), \dots, \iota_m(a_{m-1}), b \rangle \in \mathfrak{N}$.

Consider the function $\iota^* \supset \iota_m$ with domain $\{a_0, \dots, a_{m-1}, a\}$ such that $\iota^*(a) = b$. Now, note that for any constant $t \in \mathcal{L}$,

$$t^M = a \Leftrightarrow t = y \text{ occurs in } \theta_{k-(m+1)}(x_0, \dots, x_{m-1}, y) \Leftrightarrow t^N = b.$$

Moreover, for any n -ary relation $R \in \mathcal{L}$, for every $\bar{a} \in \{a_0, \dots, a_{m-1}, a\}^n$ and $\bar{x} \in \{x_0, \dots, x_{m-1}, y\}^n$,

$$\bar{a} \in R^M \Leftrightarrow R(\bar{x}) \text{ occurs in } \theta_{k-(m+1)}(x_0, \dots, x_{m-1}, y) \Leftrightarrow \iota^*(\bar{a}) \in R^N.$$

The case of functions can be reduced to the case of relations. So, ι^* is an isomorphism. ■

Based on Lemma 2, we can prove that in urn logics every unnested formula has a set of Hintikka normal forms.

Theorem 2. *Let ϕ be an unnested formula with quantifier rank k and free-variables $x_0, \dots, x_n$. Then, for every $q \geq k$, there is a Hintikka normal form ψ equivalent to ϕ and with quantifier rank q .*

Proof. Proof by induction on ϕ . If ϕ is atomic formula, then ψ is $\bigvee \Gamma$, for

$$\Gamma = \{\theta_q(x_0, \dots, x_n) : M_{\mathfrak{M}} \models \theta_q(a_0, \dots, a_n) \wedge \phi,$$

$$\text{for some } M, \mathfrak{M} \text{ and } a_0, \dots, a_n \in M\}.$$

To verify this, first assume that, for some structure M and a set of choice sequences $\mathfrak{M}$, $M_{\mathfrak{M}} \models \phi(a_0, \dots, a_n)$. By Lemma 6, there is some state description θ_q such that $M_{\mathfrak{M}} \models \theta_q(a_0, \dots, a_n)$. Hence, $M_{\mathfrak{M}} \models \bigvee \Gamma(a_0, \dots, a_n)$, by definition of Γ .

On the other hand, assume $M_{\mathfrak{M}} \not\models \phi(a_0, \dots, a_n)$. Moreover, suppose that $M_{\mathfrak{M}} \models \theta_q(a_0, \dots, a_n)$, for some $\theta_q \in \Gamma$. Then, by definition of Γ , there is some structure N and some set of choice sequences $\mathfrak{N}$ such that $N_{\mathfrak{N}} \models \theta_q(b_0, \dots, b_n) \wedge \phi$. However, by Lemma 7, θ_q defines an isomorphism ι ;

$\{a_0, \dots, a_n\} \rightarrow \{b_0, \dots, b_n\}$ and, consequently, $M_{\mathfrak{M}} \models \phi(a_0, \dots, a_n)$, what contradicts our original hypothesis. So, $\phi \equiv \bigvee \Gamma$.

For the inductive step, fix some $q \geq k$ and assume ϕ is $\psi \wedge \gamma$. By inductive hypothesis, there are Hintikka normal forms $\bigvee \Psi$ and $\bigvee \Gamma$ of ψ and γ , respectively, with quantifier rank q . So, $\bigvee(\Psi \cap \Gamma)$ is the Hintikka normal form of ϕ . A similar reasoning shows that $\bigvee(\Psi \cup \Gamma)$ and $\bigvee(\bar{\Psi})$ are the Hintikka normal forms of $\psi \vee \gamma$ and $\neg\psi$, respectively, for $\bar{\Psi} = \{\theta_q(x_0, \dots, x_n) : \theta_q(x_0, \dots, x_n) \notin \Psi\}$.

Finally, suppose that ϕ is $\exists y\psi$. By inductive hypothesis, ψ has a Hintikka normal form $\bigvee \Gamma$ with quantifier rank $q - 1$. So, $\exists y\psi$ is equivalent to $\exists y \bigvee \Gamma$. Let $\Delta_0, \dots, \Delta_m$ be all the non-empty elements of the power set of Γ . For every $i \leq m$, the formula $\bigwedge \exists y \Delta_i \wedge \forall y \bigvee \Delta_i$ is a state description with quantifier rank q . Further, $\bigvee \Gamma$ is equivalent to $\bigvee_{i \leq m} (\bigwedge \exists y \Delta_i \wedge \forall y \bigvee \Delta_i)$. Therefore, this is a Hintikka normal form of ϕ . ■

Theorem 2 establishes that in urn logics at least unnested formulas have Hintikka normal forms. What about nested formulas? The next result shows that in urn logics it is possible to weakly generalize Theorem 2 for all formulas.

In urn logics, every formula is *equisatisfiable* with some unnested translation of it. We will not prove this fact here (see [Mendonça, 2018, p. 50] for more details), but the following example gives an illustration. For instance, consider a pair $(M, \mathfrak{M})$ satisfying the formula $f(g(x)) = c$ and let $g(x)^M = a$ and $f(a)^M = b$, for some $a, b \in M$. Let $\mathfrak{M}'$ be a variant of $\mathfrak{M}$ such that $\langle a, b \rangle \in \mathfrak{M}'$. Then, $M_{\mathfrak{M}'} \models \exists y \exists w (g(x) = y \wedge f(y) = w \wedge c = w)$. So, $f(g(x)) = c$ is equisatisfiable with the formula $\exists y \exists w (g(x) = y \wedge f(y) = w \wedge c = w)$.

Corollary 1. Let ϕ be a formula with quantifier rank k . Then, for every $q \geq k$, there is a Hintikka normal form ψ with quantifier rank q equisatisfiable with ϕ .

Proof. Consider a pair $(M, \mathfrak{M})$ that satisfies ϕ . There is a variant $\mathfrak{M}'$ of $\mathfrak{M}$ and an unnested translation ϕ' of ϕ such that $M_{\mathfrak{M}'} \models \phi'$. By Theorem 2, for any $q \geq k$, there is some Hintikka normal form ψ equivalent to ϕ' with quantifier rank q . So, ψ is equisatisfiable with ϕ . ■

4. Characterization theorems

In this section we finally show that a special class of Hintikka normal forms determines a characterization theorem for perfect urn logic. Moreover, we indicate how we can obtain an equivalent result for imperfect urn logic.

In what follows, for any atomic formula ϕ , we say that the *positive literal* ϕ occurs in some state description θ if and only if ϕ occurs in θ and $\neg\phi$ does not occur in θ .

Definition 5. A state description θ is p-consistent if and only if the following holds:

- For any term t that occurs in θ , the positive literal $(t = t)$ occurs in θ ;
- For any state description θ' with quantifier rank 0 occurring in θ , for any terms $t_0, \dots, t_n, s_0, \dots, s_n$ such that the positive literal $(t_i = s_i)$ occurs in θ' , and for every atomic formula ϕ with free-variables within $x_0, \dots, x_n$, the positive literal $\phi(t_0, \dots, t_n/x_0, \dots, x_n)$ occurs in θ' if and only if the positive literal $\phi(s_0, \dots, s_n/x_0, \dots, x_n)$ occurs in θ' ;
- For every atomic formula ϕ whose terms are free in θ , the positive literal ϕ occurs in θ if and only if $\neg\phi$ does not occur in θ .

We say that a Hintikka normal form is p-consistent if and only if every state description occurring in it is p-consistent.

In classical logic, we can prove that consistent formulas are satisfiable by showing that they define *Hintikka sets* [Hodges, 1997, pp. 40–42]. We can explore a similar strategy to show that, in perfect urn logic, all and only p-consistent Hintikka normal forms are satisfiable.

In what follows, let a *subformula chain* be a sequence of formulas $\langle \phi_0, \dots, \phi_n \rangle$ such that ϕ_j is *subformula* of ϕ_i , for every $0 \leq i < j \leq n$. We rely here on a “relaxed” concept of subformula: in particular, consider that $\neg\phi$ and $\neg\psi$ are subformulas of both $\neg(\phi \wedge \psi)$ and $\neg(\phi \vee \psi)$. Moreover, let $\phi(t/x)$ be a subformula of both $\exists x\phi$ and $\forall x\phi$, and let $\neg\phi(t/x)$ be a subformula of $\neg\exists x\phi$ and $\neg\forall x\phi$, for any term t of the considered language.

For any quantifier Q , for any subformula chain T of the form

$$\langle \phi_0, \dots, Qx\psi, \psi(t/x), \dots, \phi_n \rangle,$$

t is called a *witness* of T . The *sequence of witnesses* $\langle t_0, \dots, t_n \rangle$ of T is the collection of witnesses of T such that t_i occurs first in T than t_j , for every $0 \leq i < j \leq n$. Finally, the quantifier rank of a subformula chain T , in symbols $QR(T)$, is the total number of formulas of the form $Qx\psi$ in T .

Definition 6. For a language $\mathcal{L}$, consider a collection of sets $\{\Delta_n\}_{n \in \mathbb{N}}$ such that the following holds:

1. $(t = t) \in \Delta_0$, for every term t of $\mathcal{L}$ with complexity at most 2;
2. For every atomic formula ϕ , either $\phi \notin \Delta_0$ or $\neg\phi \notin \Delta_0$;

3. For every $n \in \mathbb{N}$, If $\phi \wedge \psi \in \Delta_n$, then $\phi, \psi \in \Delta_n$; If $\neg(\phi \wedge \psi) \in \Delta_n$, then either $\neg\phi$ or $\neg\psi$ are in Δ_n (Dual clauses can be defined for $\phi \vee \psi$ and $\neg(\phi \vee \psi)$);
4. For every $n \in \mathbb{N}$, if $\neg\neg\phi \in \Delta_n$, then $\phi \in \Delta_n$;
5. For every $n \in \mathbb{N}$, for every atomic formula ϕ with free-variables $x_0, \dots, x_m$, if there are terms $t_0, \dots, t_m, s_0, \dots, s_m$ such that $(t_i = s_i) \in \bigcup_{j \leq n} \Delta_j$, then

$$\phi(t_0, \dots, t_m/x_0, \dots, x_m) \in \Delta_n \Leftrightarrow \phi(s_0, \dots, s_m/x_0, \dots, x_m) \in \Delta_n;$$
6. For every formula ϕ that is either atomic or the negation of an atomic formula, if $\phi \in \Delta_n$, then $\phi \in \Delta_{n+1}$;
7. Assume $\exists x\phi \in \Delta_n$ and $\{T_0, \dots, T_i, \dots\}$ is the set of all subformula chains defined in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\phi$ is the last element in every T_i and $QR(T_i) = n+1$. Then, for each T_i , there is a constant $c_{\phi,i}$ of $\mathcal{L}$ such that $\phi(c_{\phi,i}/x) \in \Delta_{n+1}$ (A dual condition holds for $\neg\forall x\phi$);
8. Assume $\forall x\phi \in \Delta_n$. For every formula ψ of $\mathcal{L}$ and for every non-empty set of subformula chains $\{T_0, \dots, T_i, \dots\}$ defined in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\psi \in \Delta_n$ is the last element of each T_i , $QR(T_i) = n+1$, if there is some T_i with the same sequence of witnesses as that of some subformula chain T defined in $\bigcup_{j \leq n} \Delta_j$ with $\forall x\phi$ as its last element, then $\phi(c_{\psi,i}/x) \in \Delta_{n+1}$, for the constant $c_{\psi,i}$ of $\mathcal{L}$ defined as in clause 7 (A dual condition holds for $\neg\exists x\phi$).

A collection of sets $\bigcup_{j \leq n} \Delta_j$ such that Δ_n has only quantifier-free formulas is a p-set of formulas of $\mathcal{L}$.

The following lemma shows that p-sets are satisfiable in perfect urn logic.

Lemma 3. *For any p-set of sentences $\Delta = \bigcup_{j \leq n} \Delta_j$ of a language $\mathcal{L}$, there is a classic structure M and a set of choice sequences $\mathfrak{M}$ such that, for every $j \leq n$, for every $\phi \in \Delta_j$ and for some sequence $\langle a_0, \dots, a_{j-1} \rangle \in \mathfrak{M}$, it is the case that $M_{\mathfrak{M}}, a_0, \dots, a_{j-1} \models \phi$.*

Proof. We will build a canonical model M and a set of choice sequences $\mathfrak{M}$ for Δ . Consider the set C of closed terms of $\mathcal{L}$ and a partition $\Pi(C)$ such that, for every $\llbracket t \rrbracket \in \Pi(C)$, $s \in \llbracket t \rrbracket$ if and only if $(t = s) \in \Delta$.

Let M be the following interpretation function:

- For every constant $c \in \mathcal{L}$, $c^M = \llbracket c \rrbracket$;
- For every m -ary relation $R \in \mathcal{L}$, $R^M = \{ \langle \llbracket t_0 \rrbracket, \dots, \llbracket t_{m-1} \rrbracket \rangle : R(t_0, \dots, t_{m-1}) \in \Delta \}$ (An analogous condition holds for m -ary functions of $\mathcal{L}$).

Let $M = (\Pi(C),^M)$. $\mathfrak{M}$ can be defined as follows:

- $\mathfrak{M}_0 = \{ \llbracket t \rrbracket : \text{for some formula } \phi \text{ of } \mathcal{L}, \exists x \phi \in \Delta_0 \text{ and } \phi(t/x) \in \Delta_1 \}$;

Now, remember that, by definition of p-set, for every formula of the form $\exists x \phi \in \Delta_j$, the set of subformula chains $\{T_0, \dots, T_i, \dots\}$ defined in $\bigcup_{k \leq j} \Delta_k$ such that $QR(T_i) = j + 1$ which have $\exists x \phi$ as their last element generates a set of constants $\{c_{\phi,0}, \dots, c_{\phi,i}, \dots\}$ such that $\phi(c_{\phi,i}/x) \in \Delta_{j+1}$. Hence:

- For any $\llbracket \bar{a} \rrbracket \in \mathfrak{M}_j$, let $B(\llbracket \bar{a} \rrbracket) = \{ \llbracket c_{\phi,i} \rrbracket : \bar{a} \text{ is the sequence of witnesses of some subformula chain } T_i \text{ defined in } \bigcup_{k \leq j} \Delta_k \text{ with } \exists x \phi \text{ as its last element, for some formula } \phi \text{ of } \mathcal{L} \}$. Based on this, $\mathfrak{M}_{j+1}$ is straightforwardly defined.

The lemma is a consequence of the following claim:

(**) For every $j \leq n$, for any formula ϕ with free-variables $x_0, \dots, x_m$ and for any closed terms $t_0, \dots, t_m$ of $\mathcal{L}$, the following holds:

- If $\phi(t_0, \dots, t_m/x_0, \dots, x_m) \in \Delta_j$, then $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \phi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$, for every $\llbracket \bar{s} \rrbracket \in \mathfrak{M}_{j-1}$ such that $\bar{s}$ is the sequence of witnesses of some subformula chain T defined in $\bigcup_{k \leq j} \Delta_k$ with $\phi(t_0, \dots, t_m/x_0, \dots, x_m)$ as its last element;
- If $\neg \phi(t_0, \dots, t_m/x_0, \dots, x_m) \in \Delta_j$, then $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \not\models \phi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$, for every $\llbracket \bar{s} \rrbracket \in \mathfrak{M}_{j-1}$ such that $\bar{s}$ is the sequence of witnesses of some subformula chain T defined in $\bigcup_{k \leq j} \Delta_k$ with $\neg \phi(t_0, \dots, t_m/x_0, \dots, x_m)$ as its last element.

We can prove (**) by induction on ϕ . If ϕ is atomic, then the property follows by definition of M and by clauses 2 and 5 of Definition 6.

Assume ϕ is $\psi \wedge \gamma$ and $\psi \wedge \gamma \in \Delta_j$. Moreover, assume that $\bar{s}$ is the sequence of witnesses of a subformula chain T defined in $\bigcup_{k \leq j} \Delta_k$ with ϕ as its last element.

Then, $\psi, \gamma \in \Delta_j$ and both formulas extend T to two other subformula chains that have the same sequence of witnesses. Therefore, by inductive hypothesis,

$M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \psi \wedge \gamma(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$. On the other hand, assume $\neg(\psi \wedge \gamma) \in \Delta_j$. Then, either $\neg\psi$ or $\neg\gamma$ are in Δ_j . Without loss of generality, suppose $\neg\psi \in \Delta_j$. So, by inductive hypothesis, $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \neg(\psi \wedge \gamma)(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$. Based on a similar argument we can verify that the property holds for $\psi \vee \gamma$ and $\neg(\psi \vee \gamma)$.

Assume ϕ is $\neg\psi$. If $\phi \in \Delta_j$, then, by inductive hypothesis,

$$M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \phi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket).$$

On the other hand, if $\neg\phi \in \Delta_j$, then $\psi \in \Delta_j$ and, by inductive hypothesis, $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \not\models \phi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$.

Assume ϕ is $\exists x\psi$ and $\phi \in \Delta_j$. Then, $\psi(c_{\psi,i}/x) \in \Delta_{j+1}$. By inductive hypothesis, $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket, \llbracket c_{\psi,i} \rrbracket \models \psi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket, \llbracket c_{\psi,i} \rrbracket)$. Given that $\llbracket c_{\psi,i} \rrbracket \in B(\llbracket \bar{s} \rrbracket)$, then $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \exists x\psi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$.

Finally, assume ϕ is $\forall x\psi$ and $\phi \in \Delta_j$. Fix some formula γ such that there is some subformula chain T defined in $\bigcup_{k \leq j} \Delta_k$ with $\exists x\gamma$ as its last element and whose sequence of witnesses is $\bar{s}$. Then, $\psi(c_{\gamma,i}/x) \in \Delta_{j+1}$. By inductive hypothesis,

$$M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket, \llbracket c_{\gamma,i} \rrbracket \models \psi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket, \llbracket c_{\gamma,i} \rrbracket).$$

By definition of $B(\llbracket \bar{s} \rrbracket)$, these are all the terms that need to be considered in order to conclude that $M_{\mathfrak{M}}, \llbracket \bar{s} \rrbracket \models \forall x\psi(\llbracket t_0 \rrbracket, \dots, \llbracket t_m \rrbracket)$. Based on a similar argument we can verify that the property holds for $\neg\forall x\psi$ and $\neg\exists x\psi$. ■

Lemma 4. *For every p -consistent state description θ with quantifier rank $n+1$ there is a p -set $\Delta = \bigcup_{j \leq n+1} \Delta_j$ such that $\theta \in \Delta_0$.*

Proof. Let $\mathcal{L}$ be a finite language of θ . By definition, θ is of the form

$$(\bigwedge \exists x_0 \Gamma) \wedge (\forall x_0 \bigvee \Gamma),$$

for some set Γ of game normal forms. So, let Δ_0 be the set that collects all the conjuncts of θ plus all identities $t = t$ such that t is a term of complexity at most 2 of $\mathcal{L}$.

In order to define Δ_1 , consider an extension $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi \in \Gamma\}$. Based on this extension, let

$$\begin{aligned} \Delta_1 = & \{\phi(c_\phi/x_0) : \phi \in \Gamma\} \cup \{\bigvee \Gamma(c_\phi/x_0) : \phi \in \Gamma\} \cup \\ & \cup \{t = t : t = t \in \Delta_0\} \cup \{\psi : \psi \text{ is either a conjunct of } \phi(c_\phi/x_0) \text{ or} \\ & \text{is a disjunct of } \bigvee \Gamma(c_\phi/x_0), \text{ for some } \phi \in \Gamma\}. \end{aligned}$$

For any $\phi' \in \Gamma$, every $\phi(c_{\phi'}/x_0) \in \Delta_1$ is a game normal form of the form $(\bigwedge \exists x_1 \Gamma_\phi) \wedge (\forall x_1 \bigvee \Gamma_\phi)$. Assume $\phi_0, \dots, \phi_m$ are all such formulas. In order to define Δ_2 , consider a new extension $\mathcal{L}_2 = \mathcal{L}_1 \cup \{c_\psi : \psi \in \Gamma_{\phi_i}, 0 \leq i \leq m\}$.

Let $\Sigma' = \{\psi(c_\psi/x_1) : \psi \in \Gamma_{\phi_i}, 0 \leq i \leq m\}$. For every $0 \leq i \leq m$, let T_i be the subformula chain defined in $\Delta_0 \cup \Delta_1$ with $\forall x_1 \bigvee \Gamma_{\phi_i}$ as its last element and let $\Sigma'' = \{\bigvee \Gamma_{\phi_i}(c_\psi/x_1) : \text{the subformula chain } T \text{ defined in } \Delta_0 \cup \Delta_1 \text{ with } \exists x_1 \psi \text{ as its last element has the same sequence of witnesses as } T_i\}$. Based on this, consider that

$$\Delta_2 = \Sigma' \cup \Sigma'' \cup \{t = t : t = t \in \Delta_1\} \cup$$

$$\cup \{\psi : \psi \text{ is either a conjunct of some element of } \Sigma' \text{ or a disjunct of some element of } \Sigma''\}.$$

The reiteration of this process $n + 1$ -times generates an adequate p-set Δ . ■

Theorem 3. *For every state description θ , θ is satisfiable in perfect urn logic if and only if θ is p-consistent.*

Proof. (Sufficiency) Assume θ is not p-consistent. Then, we must consider the following three cases:

- Case 1: for some term t of the considered language, $t \neq t$ occurs in θ . In this case, in every game $G(M, \mathfrak{M}, \theta)$ there is a winning strategy for Abelard, namely, to force Eloise to hold $t \neq t$.
- Case 2: in some state description θ' occurring in θ with $qr(\theta') = 0$, for some terms $t_0, \dots, t_n, s_0, \dots, s_n$ such that the positive literal $t_i = s_i$ occurs in θ' , for some atomic formula ϕ with free-variables $x_0, \dots, x_n$, the positive literal $\phi(t_0, \dots, t_n/x_0, \dots, x_n)$ occurs in θ' but the positive literal $\phi(s_0, \dots, s_n/x_0, \dots, x_n)$ does not occur in θ' . In this case, Abelard has the following winning strategy in a game $G(M, \mathfrak{M}, \theta)$. Consider that Eloise substitutes $t_0, \dots, t_n, s_0, \dots, s_n$ by some $a_0, \dots, a_n \in M$ during a match of $G(M, \mathfrak{M}, \theta)$. If $M_{\mathfrak{M}}, a_0, \dots, a_n \models \phi$, then Abelard enforces Eloise to hold the formula $\neg\phi(s_0, \dots, s_n/x_0, \dots, x_n)$; Otherwise, Abelard enforces Eloise to hold the formula $\phi(t_0, \dots, t_n/x_0, \dots, x_n)$.
- Case 3: for some atomic formula ϕ whose terms $t_0, \dots, t_n$ are not bounded in θ , both ϕ and $\neg\phi$ occur in θ . In this case, Abelard has the following winning strategy in any game $G(M, \mathfrak{M}, \theta)$: if $M_{\mathfrak{M}} \models \phi$, then Abelard enforces Eloise to hold $\neg\phi$; Otherwise, Abelard enforces Eloise to hold ϕ .

(Necessity) The proof is immediate from Lemmas 3 and 4. ■

Corollary 2. [Characterization theorem for perfect urn logic] A formula ϕ is satisfiable in perfect urn logic if and only if ϕ has p-consistent Hintikka normal forms.

Imperfect urn logic describes a stricter notion of satisfiability than perfect urn logic. So, finally, let us present a quick remark on how we can obtain a similar characterization theorem for the former logical system.

Essentially, in imperfect urn logic different choice n -sequences $\bar{a}$ and $\bar{b}$ have always the same choice sets, that is, $B(\bar{a}) = B(\bar{b})$. Now, if the concept of choice sequence is syntactically grasped by the notion of a sequence of witnesses, then what we need to do is to define a subclass of the set of p-consistent state descriptions such that, for any θ in this subclass, any two existential formulas $\exists x\phi$ and $\exists x\psi$ occurring in θ with the same quantifier rank should accept the same witnesses. This subset of formulas defines a notion of consistency for imperfect urn logic (for simplicity, we call this notion *i-consistency*). If we generalize this property for Hintikka normal forms (by saying that for any two state descriptions θ and θ' occurring in some Hintikka normal form, any two existential formulas $\exists x\phi$ and $\exists x\psi$ occurring in them with the same quantifier rank accept the same witnesses), then, appealing to a strategy similar to the one used above, we could show that these are all and the only formulas which are satisfiable in imperfect urn logic. Since the proof of this result is very similar to the one presented above, we let to the reader the task of proving this theorem – the reader can get more information about it in [Mendonça, 2018].

5. How have we avoided the scandal of deduction?

In this paper we have shown that we can define truth-conditional semantics for both perfect and imperfect urn logics. Furthermore, we have presented a full characterization of the sets of logical validities of these systems. In the case of perfect urn logic, every formula satisfiable in this system is equisatisfiable with a p-consistent Hintikka normal form. By its turn, in the case of imperfect urn logic, we have indicated that the satisfiable formulas are all equisatisfiable with i-consistent Hintikka normal forms. Now, we need to go back and ask again: what happens to the scandal of deduction under this new logical framework?

It is not difficult to see that traditional theory of semantic information based on urn logics in fact blocks the scandal of deduction. We can see this by noticing that classical logic defines a proper subclass of structures of urn logic, i.e., we can conceive of a classic structure M as a model whose set of choice sequences $\mathfrak{M}$ is equal to the set of all sequences of M . Now, given the fact that, in this sense, classical logic is a semantic fragment of urn logics, some formulas which are unsatisfiable in the former system become satisfiable in the latter ones. So, many classical logical truths turn out to be not valid in urn logics and, consequently, have its real informativeness acknowledged in this new theoretical context.

However, this is too much theoretical. The reader might want to see some concrete examples of classical logical validities which show themselves to be really informative in this new logical framework. In order to see one such example, we can consider again Russell's paradox, the anomalous consequence of naive set theory. As we have said before, beginning students of set theory often find it difficult to see that the notion of a set of all sets that do not contain themselves as elements is contradictory. Hence, these reasoners ignore that the sentence $\neg\exists x\forall y(y \in x \leftrightarrow y \notin y)$ is logically valid. In urn logics, we can formalize the epistemological standpoint of a person who ignores the logical validity of that sentence in the following way. Consider a classic structure Q and a set of choice sequences Ω such that $\Omega_0 = \{a\}$, $\Omega_1 = \{b\}$ and Q classically satisfies $b \in a$ and $b \notin b$. It is easy to see that $Q_\Omega \models \exists x\forall y(y \in x \leftrightarrow y \notin y)$. So, based on urn logics, the epistemological standpoint of an individual who does not know that the sentence $\neg\exists x\forall y(y \in x \leftrightarrow y \notin y)$ is logically valid can be explained by the idea that her knowledge of truth-conditions does not exclude impossible models such as Q_Ω .

Finally, let us make a few more general remarks on what more can we expect to obtain from traditional theory of semantic information in this new logical framework. Since urn logics are decidable formal systems [Olin, 1978, p. 357], with this replacement we obtain a decidable theory of semantic information. This is a theoretical advantage for the following reason: given that here we associated semantic information with the epistemological standpoint of ordinary reasoners, this conception presupposes (at least from an internalist approach to epistemology) that ordinary reasoners are able to effectively measure the semantic information of a given sentence. This presupposition is entirely met by the decidability of urn logics.

However, one question remains unanswered: given that there are at least two different systems of urn logic, which one provides the best framework for the development of a theory of semantic information? This is a difficult question. Perhaps, in order to solve this issue, we should do an empirical research on the actual conditions of the epistemological standpoint of ordinary reasoners towards logical knowledge or, even, of the knowledge associated with the semantic competence of linguistic users. Anyway, this discussion surpasses the limits of the present exposition. Provisionally, we could state that perfect urn logic, given its generality (that is, given the fact that this system semantically includes both classical logic and imperfect urn logic), is a quite appropriate framework for a theory of semantic information.

Acknowledgements. This paper is based on the PhD Thesis of the first author under the supervision of the second [Mendonça, 2018], which received the 2019 CAPES-Brazil award for Best Thesis in Philosophy. The first author is supported

by a postdoc scholarship from CAPES Brazil (project 88887.475401/2020-00). The second author has been supported by an individual research grant from CNPq Brazil (grant 307376/2018-4).

References

- Bar-Hillel, Carnap, 1953 – Bar-Hillel, Y., Carnap, R. “Semantic information”, *The British Journal for the Philosophy of Science*, 1953, Vol. 4, No. 14, pp. 147–157.
- Cresswell, 1982 – Cresswell, M.J. “Urn models: A classical exposition”, *Studia Logica*, 1982, Vol. 41, No. 2–3, pp. 109–130.
- Floridi, 2004 – Floridi, L. “Outline of a theory of strongly semantic information”, *Minds and machines*, 2004, Vol. 14, No. 2, pp. 197–221.
- Freire, 2015 – Freire, R.A. “First-order logic and first-order functions”, *Logica Universalis*, 2015, Vol. 9, No. 3, pp. 281–329.
- Hintikka, 1965 – Hintikka, J. “Distributive normal forms in first-order logic”, in: *Studies in Logic and the Foundations of Mathematics*, Elsevier, 1965, Vol. 40, pp. 48–91.
- Hintikka, 1970a – Hintikka, J. “Information, deduction, and the a priori”, *Nous*, 1970, pp. 135–152.
- Hintikka, 1970b – Hintikka, J. “Surface information and depth information”, in: *Information and inference*, Springer, 1970, pp. 263–297.
- Hodges, 1997 – Hodges, W. *A shorter model theory*. Cambridge university press, 1997.
- Mendonça, 2018 – Mendonça, B.R. *Traditional theory of semantic information without scandal of deduction*, PhD Thesis, Unicamp, 2018. <http://repositorio.unicamp.br/handle/REPOSIP/332305>.
- Olin, 1978 – Olin, P. “Urn models and categoricity”, *Journal of Philosophical Logic*, 1978, Vol. 7, No. 1, pp. 331–345.
- Rantala, 1975 – Rantala, V. “Urn models: a new kind of non-standard model for first-order logic”, in: *Game-Theoretical Semantics*, Springer, 1975, pp. 347–366.
- Tulenheimo, 2018 – Tulenheimo, T. “Independence friendly logic”, in: *The Stanford Encyclopedia of Philosophy*, ed. by E.N. Zalta, Metaphysics Research Lab, Stanford University, fall 2018 edition, 2018.