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Equality and Apartness in Bi-intuitionistic Logic

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Abstract: In the present paper we argue that a symmetric picture of the relationships between equality and apartness can be attained by considering these notions on the background of bi-intuitionistic logic instead of the usual intuitionistic logic. In particular we show that, as the intuitionistic negation of a relation of apartness is an equality, the dual-intuitionistic co-negation of an equality is a relation of apartness. At the same time, as the intuitionistic negation of equality is not an apartness, the co-intuitionistic negation of an apartness is not an equality.

Keywords: Bi-intuitionistic logic, Equality, Apartness

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1. Introduction

Bi-intuitionistic logic, introduced by [Rauszer, 1974a; Rauszer, 1974b] as “H-B logic” and also called “subtractive logic” in [Crolard, 2001], is a conservative extension of propositional intuitionistic logic obtained by the addition of a new connective: co-implication $\not\vdash$ (also referred to as “pseudo-difference”, e.g., in Rauszer’s original work, or as “subtraction”, e.g., by Crolard).

While implication relates to conjunction as follows:

$$A \wedge B \vdash C \text{ iff } A \vdash B \supset C$$

co-implication relates to disjunction as follows:

$$A \vdash B \vee C \text{ iff } A \not\vdash B \vdash C$$

Intuitively, a formula $A \not\vdash B$ can be read as “ A but not B ” or “ A excludes B ”.

In the $\{\wedge, \vee, \sim\}$ -fragment of the language of classical logic both implication and co-implication can be defined, respectively as $\sim A \vee B$ and $A \wedge \sim B$. On the other hand, in intuitionistic logic implication is independent from the other connectives and a well-known result is that also co-implication is undefinable in terms of the intuitionistic connectives, see e.g., Theorem 5.2 in [Urbas, 1996].

As implication is the distinctive connective of intuitionistic logic, the natural habitat of co-implication is dual-intuitionistic logic. In sequent calculi, a system for intuitionistic logic can be obtained by restricting all sequents in a calculus for classical logic to at most one formula in the succedent. Similarly, a sequent calculus for dual-intuitionistic logic can be obtained by imposing the dual restriction to classical sequents: at most one formula in the antecedent.

In classical logic, logical consequence can be equivalently characterized as “forward” truth-preservation or as “backwards” falsity preservation in all interpretations: a sequent $\Gamma \Rightarrow \Delta$ is classically valid iff for every interpretation, if all formulas in Γ are true, at least one formula in Δ is true; or equivalently iff for every interpretation, if all formulas in Δ are false at least one formula in Γ is false.

Informally, intuitionistic logic can be thought of as the result of replacing the classical notion of truth with the constructive notion of proof, so that $\Gamma \Rightarrow A$ is intuitionistically valid iff A is provable whenever all formulas in Γ are provable. As argued by the second author [Tranchini, 2012], dual-intuitionistic logic can be thought as the result of replacing the classical notion of falsity with a constructive notion of *refutation* or *disproof*. Accordingly $A \Rightarrow \Gamma$ is dual-intuitionistically valid iff A is refutable whenever all formulas in Γ are (here refutation is understood as a primitive notion, not to be defined in terms of some object-language negation operator).

The constructive nature of the notions of proof and refutation induce the rejection of certain classically valid principles in each of the two logics. The rejection of these principle is rewarded by stronger meta-theoretical properties, such as the disjunction property of intuitionistic logic (if $A \vee B$ is intuitionistically provable either A or B is) and its dual-intuitionistic counterpart (if $A \wedge B$ is dual-intuitionistically refutable either A or B is), which represent the cornerstone of their constructive reading.

One may expect that bi-intuitionistic logic could be given an informal interpretation in terms of both proofs and refutations. This is, however, no obvious task, due to the fact that the disjunction property and its dual do not hold in bi-intuitionistic logic.

A further difficulty concerns the proper formulation of the duality underlying bi-intuitionistic logic. The duality between intuitionistic and dual-intuitionistic logic can be made precise by defining a mapping $()^*$ from the language of bi-intuitionistic logic to itself so that $A^* = A$ for atomic proposition, and

$$\begin{aligned} (\top)^* &= \perp \\ (\perp)^* &= \top \\ (A \wedge B)^* &= A^* \vee B^* \\ (A \vee B)^* &= A^* \wedge B^* \\ (A \supset B)^* &= B^* \not\supset A^* \\ (A \not\supset B)^* &= B^* \supset A^* \end{aligned}$$

The duality amounts to the fact that $\Gamma \Rightarrow A$ is intuitionistically valid iff $A^* \Rightarrow \Gamma^*$ is dual-intuitionistically valid and $A \Rightarrow \Gamma$ is dual-intuitionistically valid iff $\Gamma^* \Rightarrow A^*$ is intuitionistically valid. This duality extends to bi-intuitionistic logic itself, so that $\Gamma \Rightarrow \Delta$ is bi-intuitionistically valid iff $\Delta^* \Rightarrow \Gamma^*$ is.

The clauses for the connectives may suggest the following informal reading of the duality: a proof of A is a refutation of A^* and viceversa. This is however, incompatible with the base clause: whereas $()^*$ relates pairs of (distinct) connectives, the atomic propositions are the dual of themselves and this blocks the possibility of informally reading the duality as exchanging the role of proofs and refutations (for similar reasons [Crolard, 2001] refers to $()^*$ as a “pseudo-duality”, rather than a genuine duality).

A natural, although so far unexplored, way of solving this second difficulty is that of considering the duality between theories rather than purely logical systems and by introducing the dual of each primitive notion used in the formalization of a given theory.

The present paper aims to be a preliminary investigation in this direction. In particular, we consider one of the most elementary theories, that of equality, and address the question as to whether the notion of apartness—well-investigated in constructive mathematics—can be taken to play the role of the dual of equality.

In constructive mathematics, the relation of apartness $a \neq b$ is a “positive” counterpart of the negative notion of inequality $\neg a = b$. In intuitionistic logic, $\neg A$ is short for $A \supset \perp$ and hence $\neg a = b$ means that the assumption $a = b$

leads to a contradiction. By contrast, two real numbers a and b are apart, $a \neq b$, when there is a third one c measuring the distance of b from a on the real line. On the constructive reading of the existential quantifier, $a \neq b$ is a stronger claim than just $\neg a = b$, since the latter does not imply that the distance between a and b on the real line can be effectively computed.

In fact, given intuitionistic logic, one *cannot define apartness* by negating equality, since the relation so defined fails to satisfy some characteristic principles of apartness. At the same time, the intuitionistic negation of apartness is a relation satisfying reflexivity, symmetry and transitivity, that is one *can define a notion of equality* as negated apartness. The defined notion, however, is not just an equivalence relation, but an equivalence relation which is also stable, i.e., it satisfies the law of double negation elimination.

Thus whereas on the background of classical logic the notions of equality and apartness are perfectly symmetric (in classical mathematics, two numbers are apart iff they are not equal and they are equal iff they are not apart), this is definitely not the case on the background of intuitionistic logic.

This asymmetry may suggest that apartness is not the best candidate to act as the dual of equality. However, once the base clause of $()^*$ is replaced with the clauses

$$(x = y)^* = x \neq y \qquad (x \neq y)^* = x = y$$

the relation between the axioms of the theories of equality and apartness is the same as the one observed above (i.e. if $\Gamma \Rightarrow \Delta$ is an axiom of equality, $\Delta^* \Rightarrow \Gamma^*$ is an axiom of apartness and vice versa).

Moreover, the bi-intuitionistic setting offers a natural way to remedy the asymmetry between identity and apartness by considering two further notions besides equality, apartness and their intuitionistic negations, namely their dual-intuitionistic co-negations. In dual- and bi-intuitionistic logic one can define a unary connective called co-negation $\neg$ using $\not\vdash$ and $\top$, by taking $\neg A$ as short for $\top \not\vdash A$. Co-negation is the dual of intuitionistic negation, i.e. $\neg A = (\neg A)^*$ and $\neg A = (\neg A)^*$.

In the present paper we will show that, as the intuitionistic negation of a relation of apartness is an equality, the co-negation of an equality is a relation of apartness. At the same time, as the intuitionistic negation of equality is not an apartness, the co-intuitionistic negation of an apartness is not an equality.

Although the results presented do not exhaust the possibilities of investigating equality and apartness in the context of bi-intuitionistic logic, we believe that suggest so far unexplored, lines of research. In particular, they demonstrate that bi-intuitionistic logic is not only interesting as a logical system, but that it can be fruitfully applied to the study of mathematical theories.

The paper is structured as follows. In Section 2. we present the semantics and proof theory of bi-intuitionistic logic and in Section 3. we introduce the theories of equality and apartness that will be discussed in the present paper. In Section 4. we summarize the results concerning the intuitionistic theory of equality and apartness as well as a strengthening of the former (the theory of stable equality) and a weakening of the latter (here called the theory of weak apartness). Although most results of this section are not new, they are here embedded in a systematic picture and clearly formulated as (in some cases faithful) interpretations of the above mentioned theories into each other. In Section 5. we consider the four theories above on the background of bi-intuitionistic logic and we show how co-negation allows to establish further relationships between them. The resulting picture, however, does not seem to suggest a real symmetry between $=$ and $\neq$ in bi-intuitionistic logic. In Section 6., we dispel this impression by considering two further theories (a weakening of the theory of equality, that we call the theory of weak equality; and a strengthening of that of apartness, the theory of co-stable apartness) and we show how intuitionistic negation and dual-intuitionistic co-negation allow to interpret each theory into any other. In Section 7. we discuss the significance of the results presented and indicate several directions along which the present work can be further developed.

2. Preliminaries

Assumed countably many individual variables, to be indicated with $x, y, z \dots$, let $\mathcal{L}$ be the language defined by the following grammar (we indicate formulas of $\mathcal{L}$ with $A, B, C \dots$):

$$\mathcal{L} ::= x = y \mid x \neq y \mid (A \wedge B) \mid (A \vee B) \mid (A \supset B) \mid A \not\supset B \mid \perp \mid \top$$

The negation $\neg A$ of a formula A is defined as $A \supset \perp$ and its co-negation $\neg A$ as $\top \not\supset A$. We indicate with $\mathcal{L}^=$ and $\mathcal{L}^\neq$ (respectively) the $\neq$ -free and $=$ -free fragments of $\mathcal{L}$, and with $\mathcal{L}^{i=}$ and $\mathcal{L}^{i\neq}$ the $\not\supset$ -free fragments of the latter languages.

A *sequent over $\mathcal{L}$* is an expression of the form $(\Gamma \Rightarrow \Delta)$, where Γ and Δ are multisets of formulas of $\mathcal{L}$, called the antecedent and succedent of the sequent, respectively. Outermost parenthesis will be mostly omitted. We indicate with $\mathcal{S}(\mathcal{L})$ the set of sequents over $\mathcal{L}$. We use similar notions and notation for the fragments of $\mathcal{L}$ introduced above.

The semantics of $\mathcal{L}$ and its fragments is based on Kripke models. A bi-intuitionistic Kripke frame consists of a non-empty set K of worlds $\alpha, \beta, \gamma, \dots$

pre-ordered by $\leq$ and a set D of objects $a, b, c, \dots$ called domain. A bi-intuitionistic Kripke model M for $\mathcal{L}$ is a bi-intuitionistic Kripke frame equipped with an interpretation function I assigning in each world α two relations $=_\alpha$ and $\neq_\alpha$ on D to the symbols $=$ and $\neq$, respectively. It is assumed that if $\alpha \leq \beta$ and $a =_\alpha b$, then $a =_\beta b$ and if $\alpha \leq \beta$ and $a \neq_\alpha b$, then $a \neq_\beta b$. Sometimes the subscript in $=_\alpha$ and $\neq_\alpha$ will be omitted.

An assignment φ maps variables to elements in D . We define inductively what it means for a formula A of $\mathcal{L}$ to hold at a possible world α with respect to an assignment φ , in symbols $\alpha \Vdash^\varphi A$.

$$\begin{aligned}
& \alpha \nVdash^\varphi \perp \\
& \alpha \Vdash^\varphi \top \\
& \alpha \Vdash^\varphi x = y \quad \text{iff} \quad \varphi(x) =_\alpha \varphi(y) \\
& \alpha \Vdash^\varphi x \neq y \quad \text{iff} \quad \varphi(x) \neq_\alpha \varphi(y) \\
& \alpha \Vdash^\varphi A \wedge B \quad \text{iff} \quad \alpha \Vdash^\varphi A \text{ and } \alpha \Vdash^\varphi B \\
& \alpha \Vdash^\varphi A \vee B \quad \text{iff} \quad \alpha \Vdash^\varphi A \text{ or } \alpha \Vdash^\varphi B \\
& \alpha \Vdash^\varphi A \supset B \quad \text{iff} \quad \beta \Vdash^\varphi B \text{ for all } \beta \geq \alpha \text{ s.t. } \beta \Vdash^\varphi A \\
& \alpha \Vdash^\varphi A \not\supset B \quad \text{iff} \quad \beta \nVdash^\varphi B \text{ for some } \beta \leq \alpha \text{ s.t. } \beta \Vdash^\varphi A
\end{aligned}$$

It is easily verified that if $\alpha \leq \beta$ and $\alpha \Vdash^\varphi A$ then $\beta \Vdash^\varphi A$. Moreover, observe that $\alpha \Vdash^\varphi \neg A$ iff $\beta \nVdash^\varphi A$, for all $\beta \geq \alpha$; and that $\alpha \Vdash^\varphi \neg A$ iff $\beta \nVdash^\varphi A$ for some $\beta \leq \alpha$.

We can now define the notion of validity in a class of models. A sequent holds at α with respect to φ , written $\alpha \Vdash^\varphi (\Gamma \Rightarrow \Delta)$, when for all $A \in \Gamma$, if $\alpha \Vdash^\varphi A$, then $\alpha \Vdash^\varphi B$, for some $B \in \Delta$. A sequent $(\Gamma \Rightarrow \Delta)$ is valid in a model M , written $M \Vdash (\Gamma \Rightarrow \Delta)$, when $\alpha \Vdash^\varphi (\Gamma \Rightarrow \Delta)$ for all $\alpha \in K$ and φ . Notice that the notion of validity in a model given here is *local*, see, for a discussion [Goré et al., 2020]. A sequent is valid in a class of models $\mathcal{C}$, indicated as $\mathcal{C} \Vdash (\Gamma \Rightarrow \Delta)$, when $M \Vdash (\Gamma \Rightarrow \Delta)$ for all $M \in \mathcal{C}$. Opportune restrictions of these notions to the various fragments of $\mathcal{L}$ will be used throughout.

By a theory over $\mathcal{L}$ we understand a subset of $\mathcal{S}(\mathcal{L})$ and similarly for its fragments. We shall conveniently use Gentzen calculi to describe theories, so that a theory will be identified with the set obtained by closing under the rules of inference the set of initial sequents of the calculus. In the present paper we will be almost exclusively concerned with three theories over $\mathcal{L}^{i=}$ and three theories over $\mathcal{L}^{i\neq}$. We describe them as extensions of a basic sequent calculus for bi-intuitionistic logic with non-logical initial sequents corresponding to the properties of equality and apartness. The underlying logical calculus **G** consists of the initial sequents and logical rules given in Table 1 (this is in fact the system **LBJ**₁ of [Kowalski et al. 2017]).

$\Rightarrow \top$	$A \Rightarrow A$	$\perp \Rightarrow$
$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} L\wedge^1$	$\frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} L\wedge^2$	$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} R\wedge$
$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} L\vee$	$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} R\vee^1$	$\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} R\vee^2$
$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \supset B, \Gamma, \Sigma \Rightarrow \Delta, \Pi} L\supset$	$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B} R\supset$	
$\frac{A \Rightarrow \Delta, B}{A \not\Rightarrow B} L\neg$	$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A \not\Rightarrow B} R\neg$	
$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW$	$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} RW$	$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC$
		$\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$
		$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} cut$

Table 1. The sequent calculus G

$\Rightarrow x = x$	(Reflexivity)
$\text{weq} := \begin{cases} \neg x = y \Rightarrow \neg x = z, \neg z = y & \text{(Co-negative Co-transitivity)} \\ \neg x = y \Rightarrow \neg y = x & \text{(Co-negative symmetry)} \end{cases}$	
$\Rightarrow x = x$	(Reflexivity)
$\text{eq} := \begin{cases} x = z, z = y \Rightarrow x = y & \text{(Transitivity)} \\ x = y \Rightarrow y = x & \text{(Symmetry)} \end{cases}$	
$\text{eq} \cup$	
$\text{seq} := \begin{cases} \neg \neg x = y \Rightarrow x = y & \text{(Stability)} \end{cases}$	
$x \neq x \Rightarrow$	(Irreflexivity)
$\text{wap} := \begin{cases} \neg x \neq z, \neg z \neq y \Rightarrow \neg x \neq y & \text{(Negative transitivity)} \\ \neg x \neq y \Rightarrow \neg y \neq x & \text{(Negative symmetry)} \end{cases}$	
$x \neq x \Rightarrow$	(Irreflexivity)
$\text{ap} := \begin{cases} x \neq y \Rightarrow x \neq z, z \neq y & \text{(Co-transitivity)} \\ x \neq y \Rightarrow y \neq x & \text{(Symmetry)} \end{cases}$	
$\text{ap} \cup$	
$\text{sap} := \begin{cases} x \neq y \Rightarrow \neg \neg x \neq y & \text{(Co-stability)} \end{cases}$	

Table 2. The sets of initial sequents yielding the different theories to be considered

We observe that by the definition of (dual-)intuitionistic (co-)negation, the following rules are derivable:

$$\begin{array}{c} \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} L_{\neg} \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A} R_{\neg} \\[10pt] \frac{\Rightarrow \Delta, A}{\neg A \Rightarrow \Delta} L_{\neg} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} R_{\neg} \end{array}$$

We indicate with superscripts the restriction of $\mathbf{G}$ to the corresponding fragments of $\mathcal{L}$ (so that e.g. $\mathbf{G}^{i=}$ is the restriction of $\mathbf{G}$ to $\mathcal{L}^{i=}$).

3. Six theories and their duality

We will consider theories obtained by extending (fragments of) $\mathbf{G}$ with the sets of initial sequents schematically depicted in Table 2. For each such theory $\mathbf{T}$, we write $\mathbf{T} \vdash (\Gamma \Rightarrow \Delta)$ iff the sequent belongs to the theory, i.e., it is derivable in the sequent calculus describing $\mathbf{T}$.

In particular, we call the result of adding to $\mathbf{G}^=$ the set of initial sequents **eq** (resp. **weq/seq**) the *bi-intuitionistic theory of (resp. weak/stable) equality*, to be indicated with **EQ** (resp. **WEQ/SEQ**).

Similarly, we call the result of adding to $\mathbf{G}^{\neq}$ the set of initial sequents **ap** (resp. **wap/sap**) the *bi-intuitionistic theory of (resp. weak/stable) apartness*, to be indicated with **AP** (resp. **WAP/SAP**).

Note that, given the rules for dual-intuitionistic co-negation, transitivity and symmetry imply co-negative co-transitivity and co-negative symmetry (respectively); and that, given the rules of intuitionistic negation, co-transitivity and symmetry imply negative transitivity and negative symmetry (respectively). Hence $\mathbf{WEQ} \subset \mathbf{EQ} \subset \mathbf{SEQ}$ and $\mathbf{WAP} \subset \mathbf{AP} \subset \mathbf{SAP}$.

We will refer to the restrictions of **EQ**, (respectively **SEQ**) and **AP** (respectively **WAP**) to the languages $\mathcal{L}^{i=}$ and $\mathcal{L}^{i\neq}$ as the intuitionistic theories of (resp. stable) equality and (resp. weak) apartness, to be indicated with $\mathbf{EQ}^i$, (resp. $\mathbf{SEQ}^i$) and $\mathbf{AP}^i$ (resp. $\mathbf{WAP}^i$). These four theories have been extensively investigated in the literature, and the intuitionistic theory of weak apartness is sometimes called “negative equality” in [Negri et al., 2001] or “defined equality” in [Negri, 1999].¹

From the perspective of bi-intuitionistic logic, the set of axioms **eq** and **ap** suggests that the relationship between $=$ and $\neq$ should be the same as the one between $\wedge$ and $\vee$ and $\supset$ and $\not\supset$ that we described in the introduction as a duality. To spell it out properly we consider a further theory on the full

¹Our terminological choice has the only purpose of making easier for the reader to remember which theories are based on the language $\mathcal{L}^{i\neq}$ and which are based on $\mathcal{L}^{i=}$.

language $\mathcal{L}$, obtained by extending $\mathbf{G}$ with both **eq** and **ap**, to be referred to as **EA**. Let $()^* : \mathcal{L} \mapsto \mathcal{L}$ be defined as described in the introduction, namely:

$$\begin{aligned} (x = y)^* &:= x \neq y & (x \neq y)^* &:= x = y \\ (\top)^* &:= \perp & (\perp)^* &:= \top \\ (A \wedge B)^* &:= A^* \vee B^* & (A \vee B)^* &:= A^* \wedge B^* \\ (A \supset B)^* &:= B^* \not\supset A^* & (A \not\supset B)^* &:= B^* \supset A^* \end{aligned}$$

and let $(\Gamma \Rightarrow \Delta)^* := (\Delta^* \Rightarrow \Gamma^*)$, where Σ^* is the multiset of all A^* such that $A \in \Sigma$. The following holds:

Theorem 1.

$$\mathbf{EA} \vdash (\Gamma \Rightarrow \Delta) \quad \text{iff} \quad \mathbf{EA} \vdash (\Gamma \Rightarrow \Delta)^*$$

Proof. The proposition is established by induction on the derivation of $\Gamma \Rightarrow \Delta$, by constructing a derivation of $(\Gamma \Rightarrow \Delta)^*$ that we call *the dual of the given derivation* of $\Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is an initial sequent it is easily verified that $(\Gamma \Rightarrow \Delta)^*$ is an initial sequent as well. If the derivation of $\Gamma \Rightarrow \Delta$ ends with an application of, e.g., $R\supset$ it suffice to apply the induction hypothesis to the immediate sub-derivation and the dual of the original derivation can be obtained by opportunely applying $L\not\supset$. The other cases are treated analogously, by exchanging an application of a left/right operational rules by an application of the right/left rule for the dual connective (in the case of structural rules it is enough to exchange left/right). ■

As suggested in the introduction, in bi-intuitionistic logic the duality can be informally understood as the possibility of obtaining meaning explanation for the dual of a give formula by replacing the notions of proof and refutation in the meaning explanations of the original formula.² For example, the proof-conditions of $A \supset B$ can be expressed by saying that a proof of $A \supset B$ is a method to transform any proof of A into a proof of B ; and the refutation-conditions of $A^* \not\supset B^*$ can be expressed by saying that a refutation of $A^* \not\supset B^*$ is a method to transform any refutation of B^* into a refutation of A^* . In particular, a proof of $\neg A$ is a method to transform proofs of A into proofs of $\perp$ (which is by definition the proposition of which there is no proof); dually a refutation of $\neg A^*$ is a method to transform refutations of A^* into refutations of $\top$ (which is by definition the proposition of which there is no refutation).

²However, as observed in the introduction, due to the fact that in bi-intuitionistic logic both disjunction property and its dual fail, it is not wholly clear how should the notions of proof of a disjunction and of refutation of a conjunction be informally characterized so as to fit the bi-intuitionistic setting. Addressing this additional difficulty goes beyond the scope of the present paper.

Proposition 1 suggests to extend this informal interpretation to the case of $=$ and $\neq$, so that as we can read e.g., transitivity as saying that given proofs of $x = z$ and $z = y$ we can construct a proof of $x = z$, we can read co-transitivity as saying that given refutations of $x \neq z$ and $z \neq y$ then we can construct a refutation of $x \neq y$.

As the results of Section 4. show, the set of initial sequents **seq** and **wap** naturally aroused by studying the relationship between $=$ and $\neq$ using intuitionistic negation. The two sets of initial sequents **weq** and **sap** are motivated by considerations of duality: as the reader can easily verify, Proposition 1 holds if one replaces the theory **EA** with either of the two theories that one obtains by extending **G** with either **weq** and **wap**, or with **seq** and **sap**. The notion of dual of a derivation, as defined in the proof of Proposition 1 extends to these further two theories as well. We observe that by construction, if the derivation of $\Gamma \Rightarrow \Delta$ is a derivation in **EQ** (resp. **AP**), its dual is a derivation in **AP** (resp. **EQ**), and similarly for **WEQ** and **WAP** and for **SEQ** and **SAP**.

In this case as well, by duality we obtain an informal interpretation of e.g. co-stability as warranting that from a refutation of $\neg\neg A$ one can obtain a refutation of A (thus dualizing the informal reading of stability), and similarly for co-negative co-transitivity and co-negative symmetry.

To establish the results below we will rely on the soundness of these theories with respect to certain classes of bi-intuitionistic models. In particular, we indicate with $\mathcal{WEL}$, $\mathcal{EL}$, $\mathcal{SEL}$, $\mathcal{WASP}$ and $\mathcal{SAP}$ the classes of models in which all sequents in **weq**, **eq**, **seq**, **wap**, **ap** and **sap** (respectively) are valid.

It is easy to see that each of the four theories considered is sound in the corresponding class of models, that is:

Theorem 2 (soundness). *The following hold:*

1. if $\mathbf{WEQ} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{WEL} \Vdash (\Gamma \Rightarrow \Delta)$;
2. if $\mathbf{EQ} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{EL} \Vdash (\Gamma \Rightarrow \Delta)$;
3. if $\mathbf{SEQ} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{SEL} \Vdash (\Gamma \Rightarrow \Delta)$;
4. if $\mathbf{WAP} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{WASP} \Vdash (\Gamma \Rightarrow \Delta)$;
5. if $\mathbf{AP} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{AP} \Vdash (\Gamma \Rightarrow \Delta)$.
6. if $\mathbf{SAP} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathcal{SAP} \Vdash (\Gamma \Rightarrow \Delta)$.

Proof. For each theory T the proof is by induction on the length of the derivation of $\Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is an initial sequent that is not of the form

$A \Rightarrow A$ the proposition is an immediate consequence of the way in which the classes of models have been defined. The remaining cases are standard. ■

If $\sigma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a translation from $\mathcal{L}_1$ to $\mathcal{L}_2$ and $(\Gamma \Rightarrow \Delta) = (A_1, \dots, A_n \Rightarrow B_1, \dots, B_m)$ we write $\sigma(\Gamma \Rightarrow \Delta)$ for $\sigma(A_1), \dots, \sigma(A_n) \Rightarrow \sigma(B_1), \dots, \sigma(B_m)$. Let X and Y be two theories over $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We say that $\sigma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is an *interpretation* of X in Y when for all $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}_1)$, if $\mathsf{X} \vdash (\Gamma \Rightarrow \Delta)$, then $\mathsf{Y} \vdash \sigma(\Gamma \Rightarrow \Delta)$. Moreover, we say that σ *preserves* X in Y when the converse holds, namely for no sequent $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}_\infty)$, $\mathsf{Y} \vdash \sigma(\Gamma \Rightarrow \Delta)$ and $\mathsf{X} \not\vdash (\Gamma \Rightarrow \Delta)$ (we call such sequents, if they exists, *counterexamples* to preservation). A preserving interpretation σ is said to be *faithful*.

We shall assume throughout that translations are structural, i.e., $\sigma(A \circ B) := \sigma(A) \circ \sigma(B)$, where $\circ$ is a binary connective, $\sigma(\perp) := \perp$ and $\sigma(\top) := \top$.

We observe the following facts (proofs are obvious and left to the reader):

Fact 1. If X and Y are two theories over the same language $\mathcal{L}$ and $\mathsf{X} \subseteq \mathsf{Y}$, then $id_{\mathcal{L}}$, (the identity function on $\mathcal{L}$) is an interpretation of X in Y , but not necessarily of Y in X . Moreover, $id_{\mathcal{L}}$ preserves X in Y iff $\mathsf{X} = \mathsf{Y}$.

Fact 2. If σ and τ are interpretations of X in Y and of Y in Z , respectively, then the composition $\tau \circ \sigma$ is an interpretation of X in Z .

Fact 3. If σ is a faithful interpretation of X in Y and τ is a non-faithful interpretation of Y in Z , $\tau \circ \sigma$ is a faithful interpretation of X in Z iff no counterexample $(\Gamma \Rightarrow \Delta)$ to the faithfulness of τ is in the range of σ , i.e., there is no $(\Sigma \Rightarrow \Theta)$ such that $\sigma(\Sigma \Rightarrow \Theta) = (\Gamma \Rightarrow \Delta)$.

We conclude this section by observing that although the theories we consider could be formulated in a first-order setting as well, most of the results below hold only for the propositional versions of the theories (see in particular footnote 3 below). Moreover, bi-intuitionistic logic is non-conservative over first-order intuitionistic logic (but only over the first-order logic of constant domains). For these reasons we decided to limit our attention to propositional theories.

4. Relating $=$ and $\neq$ with intuitionistic negation

In this section we will establish to which extent the two translations $\sigma^\neg : \mathcal{L}^{i=} \rightarrow \mathcal{L}^{i\neq}$ and $\tau^\neg : \mathcal{L}^{i\neq} \rightarrow \mathcal{L}^{i=}$

$$\sigma^\neg(x = y) := \neg x \neq y \qquad \tau^\neg(x \neq y) := \neg x = y$$

can be used to (faithfully) interpret the four theories EQ^i , SEQ^i , AP^i and WAP^i into each other.

Fig. 1. Relating EQ^i , SEQ^i , AP^i and WAP^i with intuitionistic negation

Informally, the translation $\sigma^\neg$ can be seen as an attempt to define of equality in terms of apartness and negation by taking a proof of $x = y$ to be a method to transform a proof of $x \neq y$ into a proof of $\perp$; and the translation $\tau^\neg$ can be read as an attempt to define of apartness in terms of equality and negation by taking a proof of $x \neq y$ to be a method to transform a proof of $x = y$ into a proof of $\perp$.

To enhance readability, we summarize the results to be established in Figure 1, where single/double arrows indicate non-faithful/faithful interpretations and in which we also indicate the non-faithful interpretations $id_{L^i=}$ and $id_{L^i\neq}$ of EQ^i in SEQ^i and of WAP^i in AP^i respectively (see Fact 1 above).

Theorem 3. $\tau^\neg$ is a non-faithful interpretation of WAP^i in EQ^i .

Proof. To show that $\tau^\neg$ is an interpretation of WAP^i in EQ^i , it is enough to show that $EQ^i \vdash \tau^\neg(x \neq x \Rightarrow)$, $EQ^i \vdash \tau^\neg(\neg x \neq z, \neg z \neq y \Rightarrow \neg x \neq y)$ and $EQ^i \vdash \tau^\neg(\neg x \neq y \Rightarrow \neg y \neq x)$:

$$\begin{array}{c}
 \frac{\Rightarrow x = x}{\neg x = x \Rightarrow} L_{\neg} \\
 \frac{\frac{\frac{x = z, z = y \Rightarrow x = y}{\neg x = y, x = z, z = y \Rightarrow} L_{\neg}}{\neg x = y, x = z \Rightarrow \neg z = y} R_{\neg}}{\neg x = y, x = z, \neg \neg z = y \Rightarrow} L_{\neg} \\
 \frac{\neg x = y, x = z, \neg \neg z = y \Rightarrow}{\neg x = y, \neg \neg z = y \Rightarrow \neg x = z} R_{\neg} \\
 \frac{\neg x = y, \neg \neg z = y \Rightarrow \neg x = z}{\neg \neg x = z, \neg \neg z = y \Rightarrow \neg \neg x = y} L_{\neg} \\
 \frac{\neg \neg x = z, \neg \neg z = y \Rightarrow \neg \neg x = y}{\neg \neg x = z, \neg \neg z = y \Rightarrow \neg \neg x = y} R_{\neg}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{x = y \Rightarrow y = x}{\neg y = x, x = y \Rightarrow} L_{\neg} \\
 \frac{\neg y = x, x = y \Rightarrow}{\neg y = x \Rightarrow \neg x = y} R_{\neg} \\
 \frac{\neg \neg x = y, \neg y = x \Rightarrow}{\neg \neg x = y \Rightarrow \neg \neg y = x} L_{\neg} \\
 \frac{\neg \neg x = y \Rightarrow \neg \neg y = x}{\neg \neg x = y \Rightarrow \neg \neg y = x} R_{\neg}
 \end{array}$$

To show that the interpretation is non-faithful, i.e., that $\tau^\neg$ does not preserve WAP^i in EQ^i , we consider the sequent $(\neg\neg x \neq y \Rightarrow x \neq y)$. Clearly, $\text{EQ}^i \vdash \tau^\neg(\neg\neg x \neq y \Rightarrow x \neq y)$:

$$\frac{\frac{\frac{x = y \Rightarrow x = y}{x = y, \neg x = y \Rightarrow} L_\neg}{x = y \Rightarrow \neg\neg x = y} R_\neg}{x = y, \neg\neg\neg x = y \Rightarrow} L_\neg}{\neg\neg\neg x = y \Rightarrow \neg x = y} R_\neg$$

However, $\text{WAP}^i \not\vdash (\neg\neg x \neq y \Rightarrow x \neq y)$. Consider a Kripke model M_1^i with two worlds α and β such that $\alpha \leq \beta$ and two objects a and b in D such that $a \neq_\beta b$ and $b \neq_\beta a$, i.e.,

Let $\varphi(x)$ and $\varphi(y)$ be a and b , respectively. Thus, $\alpha \Vdash^\varphi \neg\neg x \neq y$ since for all worlds $\delta \geq \alpha$ there is a world $\epsilon \geq \delta$ such that $\epsilon \Vdash^\varphi x \neq y$. But clearly $\alpha \not\Vdash^\varphi x \neq y$. Therefore $M_1^i \not\vdash \neg\neg x \neq y \Rightarrow x \neq y$. Moreover, it is easy to see that $M_1^i \in \mathcal{WAP}^i$. We only show that M_1^i satisfies the negative transitivity principle, namely $M_1^i \Vdash (\neg x \neq y, \neg y \neq z \Rightarrow \neg x \neq z)$. It suffices to show that $\alpha \Vdash^{\varphi_i} (\neg x \neq y, \neg y \neq z \Rightarrow \neg x \neq z)$, for all $i = 1, \dots, 8$ such that:

1. $\varphi_1(x), \varphi_1(y)$ and $\varphi_1(z)$ are a	5. $\varphi_5(y), \varphi_5(z)$ are a , $\varphi_5(x)$ is b
2. $\varphi_2(x), \varphi_2(y)$ are a , $\varphi_2(z)$ is b	6. $\varphi_6(x), \varphi_6(z)$ are b , $\varphi_6(y)$ is a
3. $\varphi_3(x), \varphi_3(z)$ are a , $\varphi_3(y)$ is b	7. $\varphi_7(x), \varphi_7(y)$ are b , $\varphi_7(z)$ is a
4. $\varphi_4(y), \varphi_4(z)$ are b , $\varphi_4(x)$ is a	8. $\varphi_1(x), \varphi_1(y), \varphi_1(z)$ are b

Cases (3) and (6) hold since the succedent is valid; all the remaining cases are valid since they can be obtained from initial sequents using weakening. We leave to the reader to verify that also the negative symmetry principles is valid in M_2^i . Thus, we conclude that $\mathcal{WAP}^i \not\vdash (\neg\neg x \neq y \Rightarrow x \neq y)$, hence by soundness $\text{WAP}^i \not\vdash (\neg\neg x \neq y \Rightarrow x \neq y)$. ■

To establish the next proposition we will need the following:

Lemma 1. For all $A \in \mathcal{L}^{i=}$:

1. $\text{SEQ}^i \vdash (A \Rightarrow \tau^\neg \circ \sigma^\neg(A))$
2. $\text{SEQ}^i \vdash (\tau^\neg \circ \sigma^\neg(A) \Rightarrow A)$

Proof. We establish the two claims by simultaneous induction on A :

- A is $x = y$. Clearly, $\text{SEQ}^i \vdash (x = y \Rightarrow \neg\neg x = y)$:

$$\frac{\frac{x = y \Rightarrow x = y}{x = y, \neg x = y \Rightarrow} L_{\neg}}{x = y \Rightarrow \neg\neg x = y} R_{\neg}$$

and obviously $\text{SEQ}^i \vdash (\neg\neg x = y \Rightarrow x = y)$ given the stability initial sequents.

- A is $P, \top$ or $\perp$. Obvious.
- A is $B \supset C$. We have that $\tau^{\neg} \circ \sigma^{\neg}(A) = \tau^{\neg}(\sigma^{\neg}(B \supset C)) = \tau^{\neg} \circ \sigma^{\neg}(B) \supset \tau^{\neg} \circ \sigma^{\neg}(C)$. By induction hypothesis we have that
 $\text{SEQ}^i \vdash (B \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(B))$ $\text{SEQ}^i \vdash (\tau^{\neg} \circ \sigma^{\neg}(B) \Rightarrow B)$
 $\text{SEQ}^i \vdash (C \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(C))$ $\text{SEQ}^i \vdash (\tau^{\neg} \circ \sigma^{\neg}(C) \Rightarrow C)$ and hence:

$$\frac{\frac{B \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(B) \quad \tau^{\neg} \circ \sigma^{\neg}(C) \Rightarrow C}{\tau^{\neg} \circ \sigma^{\neg}(B) \supset \tau^{\neg} \circ \sigma^{\neg}(C), B \Rightarrow C} L_{\supset}}{\tau^{\neg} \circ \sigma^{\neg}(B) \supset \tau^{\neg} \circ \sigma^{\neg}(C) \Rightarrow B \supset C} R_{\supset} \quad \frac{\frac{\tau^{\neg} \circ \sigma^{\neg}(B) \Rightarrow B \quad C \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(C)}{B \supset C, \tau^{\neg} \circ \sigma^{\neg}(B) \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(C)} L_{\supset}}{B \supset C \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(B) \supset \tau^{\neg} \circ \sigma^{\neg}(C)} R_{\supset}$$

- A is $B \wedge C$ or $B \vee C$. Similar to the previous case. ■

Corollary 1. $\tau^{\neg} \circ \sigma^{\neg}$ is a faithful interpretation of SEQ^i into itself.

Proof. From the previous lemma it is almost immediate that $\text{SEQ}^i \vdash (\Gamma \Rightarrow \Delta)$ iff $\text{SEQ}^i \vdash \tau^{\neg} \circ \sigma^{\neg}(\Gamma \Rightarrow \Delta)$ for any $(\Gamma \Rightarrow \Delta) \in \mathcal{L}^{i=}$. Let $(\Gamma \Rightarrow \Delta)$ be $(A_1, \dots, A_n, \Rightarrow B_1, \dots, B_m)$. The corollary follows by $n + m$ applications of the *Cut* rule using derivations of $A_i \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(A_i)$ and $\tau^{\neg} \circ \sigma^{\neg}(B_j) \Rightarrow B_j$ for the one direction, and derivations of $B_j \Rightarrow \tau^{\neg} \circ \sigma^{\neg}(B_j)$ and $\tau^{\neg} \circ \sigma^{\neg}(A_i) \Rightarrow A_i$ for the other direction. ■

Theorem 4. $\sigma^{\neg}$ is a faithful interpretation of SEQ^i in WAP^i .

Proof. To show that $\sigma^{\neg}$ is an interpretation of SEQ^i in WAP^i , it is enough to prove that $\text{WAP}^i \vdash \sigma^{\neg}(\Rightarrow x = x)$, $\text{WAP}^i \vdash \sigma^{\neg}(x = z, z = y \Rightarrow x = y)$, $\text{WAP}^i \vdash \sigma^{\neg}(x = y \Rightarrow y = x)$, as well as $\text{WAP}^i \vdash \sigma^{\neg}(\neg\neg x = y \Rightarrow x = y)$. The first three claims are obvious since the translation of each such sequent is an initial sequent of WAP^i . The last claim is established by a derivation of

($\neg\neg\neg x \neq y \Rightarrow \neg x \neq y$) that can be obtained by replacing $=$ with $\neq$ in the last derivation used in the proof of Proposition 3.

To show faithfulness, we need to show that, for all $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^{i=})$, if $\text{WAP}^i \vdash \sigma^\neg(\Gamma \Rightarrow \Delta)$, then $\text{SEQ}^i \vdash (\Gamma \Rightarrow \Delta)$. We reason as follows. If $\text{WAP}^i \vdash \sigma^\neg(\Gamma \Rightarrow \Delta)$, then $\text{SEQ}^i \vdash \tau^\neg \circ \sigma^\neg(\Gamma \Rightarrow \Delta)$ by Proposition 3 and hence $\text{SEQ}^i \vdash (\Gamma \Rightarrow \Delta)$ by (the faithfulness part of) Corollary 1. ■

By composing the interpretations depicted in Figure 1, we obtain further interpretations:

Corollary 2. The following hold:

1. $\tau^\neg$ is an interpretation of WAP^i in SEQ^i ;
2. $\sigma^\neg$ is an interpretation of EQ^i in WAP^i ;
3. $\sigma^\neg$ is an interpretation of EQ^i in AP^i ;
4. $\sigma^\neg$ is an interpretation of SEQ^i in AP^i .
5. $\tau^\neg \circ \sigma^\neg$ is an interpretation of SEQ^i in EQ^i ;

Proof. That the translations in (a)-(e) are interpretation of the appropriate theories follows by the above theorems and by Fact 2. ■

Theorem 5. *The interpretations in 1-3 of Corollary 2 are not faithful. The interpretations in 4 and 5 are faithful.*

Proof. We discuss the interpretations separately:

1. Faithfulness fails for the same reasons seen in the proof of Proposition 3.
- 2./3. As seen in the proof of Proposition 3, the sequent $\sigma^\neg(\neg\neg x = y \Rightarrow x = y)$ is derivable in WAP^i (and hence it is derivable in AP^i as well). Stability of equality is clearly underivable in EQ^i (to see this replace apartness with equality in M_1^i , the countermodel to the stability of apartness in the proof of Proposition 3), we conclude that $\sigma^\neg$ does not preserve EQ^i in either WAP^i or AP^i .
4. The faithfulness is a consequence of the main result of [Negri, 1999]. Negri calls a formula $A \in \mathcal{L}^{i\neq}$ *negatomic* iff all occurrences of $x \neq y$ in A are negated, and a sequent is called *negatomic* just in case it contains only negatomic formulas. Negri shows that if $\text{AP} \vdash (\Gamma \Rightarrow \Delta)$ then $\text{WAP} \vdash (\Gamma \Rightarrow \Delta)$ if $(\Gamma \Rightarrow \Delta)$ is negatomic, that is, that $\text{id}_{\mathcal{L}^{i\neq}}$ is a faithful interpretation

of WAP into AP if one restricts the attention to negatomic sequents. Since $\sigma^\neg(\Gamma \Rightarrow \Delta)$ is negatomic for any $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^{i=})$, by Fact 3 we have that $\sigma = id_{\mathcal{L}^{i=}} \circ \sigma$ is a faithful interpretation of SEQ into AP.³

5. Suppose $\text{EQ}^i \vdash \tau^\neg \circ \sigma^\neg(\Gamma \Rightarrow \Delta)$. Then, $\text{SEQ}^i \vdash \tau^\neg \circ \sigma^\neg(\Gamma \Rightarrow \Delta)$ as well. By Corollary 1 we have that also $\text{SEQ}^i \vdash (\Gamma \Rightarrow \Delta)$.⁴

■

Thus, intuitionistic negation permits to interpret all theories we are considering in the theory of apartness AP^i . However, using $\sigma^\neg$ and $\tau^\neg$ the theory of apartness AP^i cannot be interpreted into any other theory (except, of course, itself).

This follows from the following:

Theorem 6. $\tau^\neg$ is not an interpretation of AP^i in SEQ^i .

Proof. We need to show that there is a sequent $(\Gamma \Rightarrow \Delta) \in \mathcal{L}^{i=}$ such that $\text{AP}^i \vdash (\Gamma \Rightarrow \Delta)$ and $\text{SEQ}^i \not\vdash \tau^\neg(\Gamma \Rightarrow \Delta)$. Let $(\Gamma \Rightarrow \Delta)$ be an instance of co-transitivity $x \neq y \Rightarrow x \neq z, z \neq y$. Obviously, $\text{AP}^i \vdash (\Gamma \Rightarrow \Delta)$. However, its $\tau^\neg$ -translation, namely $(\neg x = y \Rightarrow \neg x = z, \neg z = y)$ is not derivable in SEQ^i . Let M_2^i be a model with three worlds α, β and γ such that $\alpha \leq \beta$ and $\alpha \leq \gamma$ and three objects a, b and c in D such that:

³We observe that Negri's result fails for the first-order version of the theory, with the negatomic sequent $\neg \forall z (\neg \neg z \neq x \vee \neg \neg z \neq y) \Rightarrow \neg x \neq y$ being a counterexample, see [van Dalen et al., 1979, p. 95].

⁴We thank one of the referees for suggesting the proof of this point.

Let $\varphi(x)$, $\varphi(y)$ and $\varphi(z)$ be a , b and c , respectively. Clearly, $\alpha \Vdash^\varphi \neg x = y$, for $\delta \not\Vdash^\varphi x = y$ for all $\delta \geq \alpha$. However, $\alpha \not\Vdash^\varphi \neg x = z$ and $\alpha \not\Vdash^\varphi \neg z = y$, for there exist two worlds $\epsilon, \epsilon' \geq \alpha$ such that $\epsilon \Vdash^\varphi x = z$ and $\epsilon' \Vdash^\varphi y = z$. Therefore, $M_2^i \not\Vdash (\neg x = y \Rightarrow \neg x = z, \neg z = y)$. Additionally, we need to show that M_2^i is in $\mathcal{SE}^i$. We leave to the reader to verify that $=$ is an equivalence relation, whereas to see that $M_2^i \Vdash (\neg \neg x = y \Rightarrow x = y)$, i.e., it satisfies also the stability principle, it is enough to show that $\alpha \Vdash^{\varphi_i} (\neg \neg x = y \Rightarrow x = y)$, for all $i = 1, \dots, 9$ such that:

1. $\varphi_1(x)$ and $\varphi_1(y)$ are a	6. $\varphi_6(x)$ is b and $\varphi_6(y)$ is c
2. $\varphi_2(x)$ is a and $\varphi_2(y)$ is b	7. $\varphi_7(x)$ is c and $\varphi_7(y)$ is a
3. $\varphi_3(x)$ is a and $\varphi_3(y)$ is c	8. $\varphi_8(x)$ is c and $\varphi_8(y)$ is b
4. $\varphi_4(x)$ is b and $\varphi_4(y)$ is a	9. $\varphi_9(x)$ and $\varphi_9(y)$ are c
5. $\varphi_5(x)$ and $\varphi_5(y)$ are b	

With respect to assignments in (1), (5) and (9), the sequent holds at α since the formula in their succedent is true at all worlds $\geq \alpha$. All other cases the sequent has a formula in the antecedent which is false at both α and β , so the whole sequent holds at α . ■

We conclude this section establishing the following.

Theorem 7. τ^- does not preserve AP^i in EQ^i .

Proof. We need to show that there exists a sequent $(\Gamma \Rightarrow \Delta)$ in $\mathcal{L}^{i \neq}$ such that $\text{EQ}^i \vdash \tau^-(\Gamma \Rightarrow \Delta)$ and $\text{AP}^i \not\Vdash (\Gamma \Rightarrow \Delta)$. Let $(\Gamma \Rightarrow \Delta)$ be the stability of apartness, nameley $\neg \neg x \neq y \Rightarrow x \neq y$. We have that $\tau^-(\Gamma \Rightarrow \Delta) := \neg \neg \neg x = y \Rightarrow \neg x = y$ which can be derived in EQ^i (see proof of Proposition 3 above) and that $(\Gamma \Rightarrow \Delta)$ is not derivable in AP^i (as shown by M_2^i , the countermodel to the stability of apartness in WAP^i given in the proof of Proposition 6, that is actually a model of AP^i). ■

Since EQ^i is stricly included in SEQ^i , we have that:

Corollary 3. The following holds:

1. τ^- is not an interpretation of AP^i in EQ^i .
2. τ^- does not preserve AP^i in SEQ^i .

5. Relating $=$ and $\neq$ with co-negation

By extending $\sigma^\neg$ and $\tau^\neg$ to $\mathcal{L}^=$ and $\mathcal{L}^\neq$ the results established in the previous section also applies to the bi-intuitionistic versions of the four theories of equality and apartness considered so far. Moreover, using $\neg$ in place of $\neg$ one can define the two further translations $\sigma^\neg : \mathcal{L}^= \rightarrow \mathcal{L}^\neq$ as follows:

$$\sigma^\neg(x = y) := \neg x \neq y \qquad \tau^\neg(x \neq y) := \neg x = y$$

Informally, one can read the translation $\sigma^\neg$ as an attempt to define equality in terms of apartness and co-negation by taking a refutation of $x = y$ to be a method to transform a refutation of $x \neq y$ into a refutation of $\top$; and the translation $\tau^\neg$ can be read as an attempt to define apartness in terms of equality and co-negation by taking a refutation of $x \neq y$ as a method to transform a refutation of $x = y$ into a refutation of $\top$.

One may expect that these two translations can help in getting a more symmetric picture of the relationship between theories of equality and theory of apartness. This is to some extent the case. As in the previous section, to improve readability we summarize the results to be established in this section in Figure 2.

Fig. 2. Relating EQ, SEQ, AP and WAP with $\neg$ and $\neg$

Theorem 8. $\tau^\neg$ is a non-faithful interpretation of AP in EQ.

Proof. To show that $\tau^\neg$ is an interpretation it is enough to prove:

$\text{EQ} \vdash \tau^\neg(x \neq x \Rightarrow)$, $\text{EQ} \vdash \tau^\neg(x \neq y \Rightarrow x \neq z, z \neq y)$ and $\text{EQ} \vdash \tau^\neg(x \neq y \Rightarrow y \neq x)$:

$$\begin{array}{c}
\frac{\Rightarrow x = x}{\neg x = x \Rightarrow} L^- \\
\frac{x = z, z = y \Rightarrow x = y}{\frac{x = z \Rightarrow x = y, \neg z = y}{\Rightarrow x = y, \neg x = z, \neg z = y} R^-} R^- \\
\frac{\neg x = y \Rightarrow \neg x = z, \neg z = y}{\neg x = y \Rightarrow \neg x = z, \neg z = y} L^- \\
\frac{y = x \Rightarrow x = y}{\Rightarrow \neg y = x, x = y} R^- \\
\frac{\Rightarrow \neg y = x, x = y}{\neg x = y \Rightarrow \neg y = x} L^-
\end{array}$$

To show that τ^- is not faithful it is enough to find a sequent $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^\neq)$ such that $\mathbf{EQ} \vdash \tau^-(\Gamma \Rightarrow \Delta)$ and $\mathbf{AP} \not\vdash (\Gamma \Rightarrow \Delta)$. Take $(\Gamma \Rightarrow \Delta)$ to be an instance of co-stability $x \neq y \Rightarrow \neg \neg x \neq y$. Now, $\tau^-(x \neq y \Rightarrow \neg \neg x \neq y) := (\neg x = y \Rightarrow \neg \neg \neg x = y)$. But $\mathbf{EQ} \vdash (\neg x = y \Rightarrow \neg \neg \neg x = y)$ (to see this take the dual of the derivation of $\neg \neg \neg x = y \Rightarrow \neg x = y$ in $\mathbf{AP}$, see the proof of Proposition 3). However, apartness is not co-stable in $\mathbf{AP}$. To see this, consider a Kripke model M_3 with two worlds α and β such that $\beta \leq \alpha$ and two objects a and b in D such that:

Let $\varphi(x)$ and $\varphi(y)$ be a and b , respectively. Clearly $\alpha \not\Vdash^\varphi \neg \neg x \neq y$, since for all worlds $\delta \leq \alpha$, namely β and α itself, there is a world $\epsilon \leq \delta$ such that $\epsilon \not\Vdash^\varphi x \neq y$. However, $\alpha \not\Vdash^\varphi x \neq y$. We leave to the reader to check that $M_3 \in \mathcal{AP}$. ■

Thus using both intuitionistic negation and dual-intuitionistic co-negation we can interpret not only $\mathbf{WAP}$ (as we already could using τ^-) but also $\mathbf{AP}$ into $\mathbf{EQ}$, and actually, by composing the different embeddings depicted in Figure 2, we have that *all* four theories can be interpreted into one another:

Corollary 4. The following hold:

1. $\sigma^- \circ \tau^-$ is an interpretation of $\mathbf{AP}$ in $\mathbf{WAP}$;
2. τ^- is an interpretation of $\mathbf{AP}$ in $\mathbf{SEQ}$.
3. τ^- is an interpretation of $\mathbf{WAP}$ in $\mathbf{EQ}$.
4. τ^- is an interpretation of $\mathbf{WAP}$ in $\mathbf{SEQ}$.

The picture is however far from symmetric. Although we can embed the theories of apartness into those of equality, none of the former ones can be faithfully interpreted into any of the latter ones (in contrast to the fact that

SEQ can be faithfully interpreted into WAP and AP). Moreover, whereas both $\tau^\neg$ and $\sigma^\neg$ interpret at least some theory based on one language into one based on the other language, we have that $\sigma^\neg$ does not interpret EQ into AP, and hence a fortiori neither EQ or SEQ into either WAP or AP:

Theorem 9. $\sigma^\neg$ is not an interpretation of EQ in AP.

Proof. We need to find a sequent $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^-)$ such that $\text{SEQ} \vdash (\Gamma \Rightarrow \Delta)$ and $\text{AP} \not\vdash \sigma^\neg(\Gamma \Rightarrow \Delta)$. Let $(\Gamma \Rightarrow \Delta)$ be $x = z, z = y \Rightarrow x = y$. Clearly, $\text{EQ} \vdash (\Gamma \Rightarrow \Delta)$. We need to show that $\text{AP} \not\vdash \sigma^\neg(x = z, z = y \Rightarrow x = y)$. Since $\sigma^\neg(x = z, z = y \Rightarrow x = y) := (\neg x \neq z, \neg z \neq y \Rightarrow \neg x \neq y)$, we only need to show that there is a Kripke model satisfying the axioms of apartness in which $\neg x \neq z, \neg z \neq y \Rightarrow \neg x \neq y$ fails. Let M_4 be a Kripke model with three worlds α, β and γ such that $\beta \leq \alpha$ and $\gamma \leq \alpha$ and three objects a, b and c in D_γ and D such that:

Again let $\varphi(x)$, $\varphi(y)$ and $\varphi(z)$ be a , b and c , respectively. Clearly, $\alpha \Vdash^\varphi \neg x \neq y$, for there exists a world $\delta \leq \alpha$, namely γ , such that $\delta \not\Vdash^\varphi x \neq y$. Similarly, $\alpha \Vdash^\varphi \neg x \neq y$. However, $\alpha \not\Vdash^\varphi \neg x \neq z$, since for all worlds $\delta \leq \alpha$, namely γ, β and α itself, $\delta \Vdash^\varphi x \neq z$. We leave to the reader to verify that $M_4 \in \mathcal{AP}$. ■

6. Enriching the picture

As intuitionistic negation does not allow to interpret AP, but only its weakening WAP into some theory of equality, one can expect that co-negation can be used to interpret only a *weakening* of the theory of equality into some theory of apartness.

Similarly, the non-faithfulness of the interpretation $\tau^\neg$ of AP into EQ witnessed by the co-stability of apartness clearly mirrors the non-faithfulness of the interpretation $\sigma^\neg$ of EQ into WAP and AP resulting by the fact that the translation of the stability of equality holds in both theories. This suggests that in order for $\tau^\neg$ to faithfully interpret some theory of apartness into a theory of equality, the former must at least validate co-stability.

Given these considerations, we take into account two further theories based on the languages $\mathcal{L}^\neg$ and $\mathcal{L}^\neq$. One, to be called SAP is the strengthening of AP obtained by adding to it initial sequents expressing the co-stability of apartness

$$x \neq y \Rightarrow \neg\neg x \neq y$$

The other one, to be called WEQ is the weakening of EQ obtained by replacing the initial sequents expressing transitivity and symmetry with initial sequents expressing what we may call “co-negative co-transitivity” and “co-negative symmetry”:

$$\neg x = y \Rightarrow \neg x = z, \neg z = y$$

$$\neg x = y \Rightarrow \neg y = x$$

Theorem 10. $\sigma^\neg$ is a non-faithful interpretation of WEQ into AP.

Proof. We have to show that $\text{AP} \vdash \sigma^\neg(\Rightarrow x = x)$, $\text{AP} \vdash \sigma^\neg(\neg x = y \Rightarrow \neg x = z, \neg z = y)$ and $\text{AP} \vdash \sigma^\neg(\neg x = y \Rightarrow \neg y = x)$. This sequents can be derived using the dual of the derivations used in the proof of Proposition 3.

To establish the non-faithfulness we reason as in Proposition 8. We need to find a sequent $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^\neg)$ such that $\text{AP} \vdash \sigma^\neg(\Gamma \Rightarrow \Delta)$ and $\text{WEQ} \not\vdash (\Gamma \Rightarrow \Delta)$. Take $(\Gamma \Rightarrow \Delta)$ to be $x = y \Rightarrow \neg\neg x = y$. Now, $\sigma^\neg(x = y \Rightarrow \neg\neg x = y) := (\neg x \neq y \Rightarrow \neg\neg\neg x \neq y)$ and $\text{AP} \vdash (\neg x \neq y \Rightarrow \neg\neg\neg x \neq y)$ (to see this take replace $=$ with $\neq$ in the dual of the derivation of $\neg\neg\neg x = y \Rightarrow \neg x = y$ in AP, see the proof of Proposition 3). However, equality is not “co-stable” in WEQ. To see this, consider a Kripke model M_4 with two worlds α and β such that $\beta \leq \alpha$ and two objects a and b in D such that:

Let $\varphi(x)$ and $\varphi(y)$ be a and b , respectively. Clearly $\alpha \not\models^\varphi \neg\neg x = y$, since for all worlds $\delta \leq \alpha$, namely β and α itself, there is a world $\epsilon \leq \delta$ such that $\epsilon \not\models^\varphi x = y$. However, $\alpha \not\models^\varphi x = y$. We leave to the reader to check that $M_4 \in \mathcal{W}\mathcal{E}\mathcal{Q}$ (in fact $M_4 \in \mathcal{E}\mathcal{Q}$). ■

To establish the next proposition we will need the following:

Lemma 2. *For all $A \in \mathcal{L}^\neq$:*

1. $\text{SAP} \vdash (A \Rightarrow \sigma^\neg \circ \tau^\neg(A))$
2. $\text{SAP} \vdash (\sigma^\neg \circ \tau^\neg(A) \Rightarrow A)$

Proof. As in the proof of Lemma 1, we establish the two claims by simultaneous induction on A :

- A is $x \neq y$. Clearly, $\text{SAP} \vdash (\neg\neg x = y \Rightarrow x = y)$:

$$\frac{\frac{x \neq y \Rightarrow x \neq y}{\Rightarrow x \neq y, \neg x \neq y} R^-}{\neg\neg x \neq y \Rightarrow x \neq y} L^-$$

and obviously $\text{SAP} \vdash (x = y \Rightarrow \neg\neg x = y)$ given the co-stability initial sequents.

- A is $P, \top$ or $\perp$. Obvious.
- A is $B \supset C$. The case is proved as the corresponding case of Lemma 1, it suffice to replace $\tau^\neg \circ \sigma^\neg$ with $\sigma^\neg \circ \tau^\neg$.
- A is $B \wedge C$ or $B \vee C$. Similar to the previous case.

■

Corollary 5. $\sigma^\neg \circ \tau^\neg$ is a faithful interpretation of SAP into itself.

Proof. The proof follows the same pattern of that of Corollary 1. ■

We can now show that:

Theorem 11. $\tau^\neg$ is a faithful interpretation of SAP into WEQ.

Proof. To show that $\tau^\neg$ is an interpretation, we have to show that $\text{WEQ} \vdash \tau^\neg(x \neq x \Rightarrow)$, $\text{WEQ} \vdash \tau^\neg(x \neq y \Rightarrow x \neq z, z \neq y)$, $\text{WEQ} \vdash \tau^\neg(x \neq y \Rightarrow y \neq x)$ and $\text{WEQ} \vdash \tau^\neg(x \neq y \Rightarrow \neg\neg x \neq y)$.

The sequent $\tau^\neg(x \neq x \Rightarrow)$ can be derived as in EQ (see proof of Proposition 8), while $\tau^\neg(x \neq y \Rightarrow x \neq z, z \neq y)$ and $\tau^\neg(x \neq y \Rightarrow y \neq x)$ are obviously derivable in WEQ, being initial sequents. Finally, one can see that $\text{WEQ} \vdash \neg x \neq y \Rightarrow \neg\neg x = y$ by taking the dual of the derivation of $\neg\neg\neg x \neq y \Rightarrow \neg x \neq y$ in WAP in the proof of Proposition 4.

To show faithfulness, we need to show that, for all $(\Gamma \Rightarrow \Delta) \in \mathcal{S}(\mathcal{L}^\neq)$, if $\text{WEQ} \vdash \tau^\neg(\Gamma \Rightarrow \Delta)$, then $\text{SAP} \vdash (\Gamma \Rightarrow \Delta)$. We reason as in the proof of Proposition 4. If $\text{WEQ} \vdash \tau^\neg(\Gamma \Rightarrow \Delta)$, then by Proposition 10, $\text{SAP} \vdash \sigma^\neg(\tau^\neg(\Gamma \Rightarrow \Delta))$ and by Corollary 5 $\text{SAP} \vdash (\Gamma \Rightarrow \Delta)$. ■

We summarize our results in Figure 3.

Fig. 3. Relating WEQ, EQ, SEQ, WAP, AP and SAP with $\neg$ and $\neg$

7. Concluding remarks

By considering theories of equality and apartness on the background of bi-intuitionistic logic we attained a fully symmetric picture of the relationship between the two notions. In particular we could show that not only the theories of equality can be embedded into those of apartness, but the other way around as well. However, we still lack a faithful interpretation of EQ into any theory of apartness, and of AP in any theory of equality. We leave this to future work.

The investigation undertaken in the present work can be further pursued into different directions.

First, notice that *a priori* there are eight possible translations, i.e., $\sigma^i \circ \tau^j$ for $i, j \in \{\neg, \neg\}$. However, only two have been explicitly considered here. Our choice was motivated primarily by the fact that these two translations are enough to provide a fully symmetric picture of the relationships among the various theories presented, but of course it would be certainly interesting to consider translations combining intuitionistic and co-intuitionistic negation. We expect that in this case, bi-intuitionistic “mixed” double-negation laws such as $\neg\neg A \Rightarrow A$ and $A \Rightarrow \neg\neg A$ will play a prominent role.⁵

Secondly, the partial faithfulness results of [Negri, 1999] for the negatomic fragment suggest the possibility of establishing similar results using co-negation.

Thirdly, it seems natural to consider further bi-intuitionistic theories beyond those considered in the present paper. On the one hand, one may consider the theory of paraconsistent apartness as based on dual-intuitionistic logic, i.e., in the implication-free fragment of $\mathcal{L}^\neq$, as is done in [Brunner, 2004]. Another possibility is to apply the idea underlying the present paper to the investigation of the theory of positive lattices [von Plato, 2001] by extending our theories with operators of join and meet.

Finally, one could investigate the notions of equality and apartness (as well as other notions) on the basis of other constructive systems based on a symmetry between positive and negative notions, such as Nelson’s logic of constructible falsity [Nelson, 1949], or Wansing’s 2-intuitionistic logic [Wansing, 2016].

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⁵Thanks to a referee for bringing this to our attention.

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