THIAGO NASCIMENTO, UMBERTO RIVIECCIO

Negation and Implication in Quasi-Nelson Logic

Thiago Nascimento

Universidade Federal do Rio Grande do Norte,
59078-970 Natal, Brazil.
E-mail: thiagnascsilva@gmail.com

Umberto Rivieccio

Universidade Federal do Rio Grande do Norte,
59078-970 Natal, Brazil.
E-mail: umberto.rivieccio@ufrn.br

Abstract: Quasi-Nelson logic is a recently-introduced generalization of Nelson’s constructive logic with strong negation to a non-involutive setting. In the present paper we axiomatize the negation-implication fragment of quasi-Nelson logic (QNI-logic), which constitutes in a sense the algebraizable core of quasi-Nelson logic. We introduce a finite Hilbert-style calculus for QNI-logic, showing completeness and algebraizability with respect to the variety of QNI-algebras. Members of the latter class, also introduced and investigated in a recent paper, are precisely the negation-implication subreducts of quasi-Nelson algebras. Relying on our completeness result, we also show how the negation-implication fragments of intuitionistic logic and Nelson’s constructive logic may both be obtained as schematic extensions of QNI-logic.

Keywords: Nelson’s constructive logic with strong negation, quasi-Nelson algebras, implication-negation subreducts, QNI-algebra, quasi-Nelson logic, algebraizable logics

For citation: Nascimento T., Rivieccio U. “Negation and Implication in Quasi-Nelson Logic”, *Logicheskie Issledovaniya / Logical Investigations*, 2021, Vol. 27, No. 1, pp. 107–123. DOI: 10.21146/2074-1472-2021-27-1-107-123

1. Introduction

Quasi-Nelson algebras are the subvariety of commutative integral bounded residuated lattices (CIBRLs, see [Galatos et al., 2007]) obtained by adding the *Nelson identity*:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y.$$

In an involutive context (i.e. if the double negation law $\sim \sim x \approx x$ is also satisfied), the Nelson identity characterizes the class of Nelson algebras, the

equivalent algebraic semantics of Nelson’s constructive logic with strong negation [Nelson, 1949]. Quasi-Nelson algebras, however, need not be involutive; indeed, it is easy to verify that every Heyting algebra (viewed as a CIBRL) satisfies the Nelson identity. Quasi-Nelson algebras are thus a common generalization of Heyting algebras and Nelson algebras.

The logic of quasi-Nelson algebras corresponds to the axiomatic extension of the *Full Lambek Calculus with exchange (e) and weakening (w)*, FL_{ew} ([Galatos et al., 2007]) obtained by adding the *Nelson Axiom*:

$$(\varphi \Rightarrow (\varphi \Rightarrow \psi)) \wedge (\sim \psi \Rightarrow (\sim \psi \Rightarrow \sim \varphi)) \Rightarrow (\varphi \Rightarrow \psi).$$

As such, quasi-Nelson logic is algebraizable, and has the class of quasi-Nelson algebras as its equivalent algebraic semantics (see [Liang, Nascimento, 2019]; for further information and motivation on quasi-Nelson algebras, see also [Rivieccio, Spinks, 2018; Rivieccio, Jansana, 2020; Rivieccio, Spinks, 2021]).

The language of (quasi-)Nelson logic includes two implication connectives, the *strong implication* ($\Rightarrow$) that satisfies the residuation property and the *weak implication* ($\rightarrow$) that enjoys the standard version of the Deduction-Detachment Theorem. Both quasi-Nelson and Nelson’s logic can be axiomatized by taking any of the two implications as primitive, defining the other through the following terms: $p \rightarrow q := p \Rightarrow (p \Rightarrow q)$ and $p \Rightarrow q := (p \rightarrow q) \wedge (\sim q \rightarrow \sim p)$. From each of the above implications (and the falsity constant 0) a negation can be defined in the standard way. In the case of Nelson’s logic, the definition $\sim p := p \Rightarrow 0$ yields the strong involutive negation, whereas $\neg p := p \rightarrow 0$ is sometimes referred to as the “intuitionistic negation”. However, in the case of quasi-Nelson logics the above terminology is less meaningful because neither of the two negations is involutive.

Like Nelson algebras, also quasi-Nelson algebras can be represented as so-called *twist-structures*, though the twist construction needs to be generalized to account for the non-involutivity of the negation (see [Rivieccio, Spinks, 2018; Rivieccio, Spinks, 2021]). Further generalizations allow us to give twist representations for some subreducts of quasi-Nelson algebras: see the recent papers [Rivieccio, Jansana, 2020; Rivieccio, 2020a; Rivieccio, 2020b; Rivieccio, 2020c]. In the present paper we are in particular interested in the representation of QNI-algebras given in [Rivieccio, Jansana, 2020, Theorem 5], corresponding to the $\{\rightarrow, \sim\}$ -subreducts of quasi-Nelson algebras. This twist construction proved to be quite useful for establishing results regarding congruences, subdirectly irreducible algebras and subvarieties of QNI-algebras; see [Rivieccio, 2020a] for more details.

As mentioned earlier, quasi-Nelson logic is algebraizable in the sense of [Blok, Pigozzi, 1989]. As a set of equivalence formulas one can take $\Delta(\varphi, \psi) = \{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\}$, and as defining equation $E(\varphi) = \{\varphi \approx \varphi \rightarrow \varphi\}$. These observations entail that the $\{\rightarrow, \sim\}$ -fragment of quasi-Nelson logic must also be algebraizable, with the same translations, with respect to the corresponding subreducts of quasi-Nelson algebras (see e.g. [Font, 2016, Proposition 3.29]). In the present paper, we introduce a finite Hilbert-style calculus that characterizes the $\{\rightarrow, \sim\}$ -fragment of quasi-Nelson logic. Indeed, we show that our calculus is algebraizable with respect to the variety of QNI-algebras introduced in [Rivieccio, Jansana, 2020].

We shall proceed in the following way. In Section 2. we introduce QNI-algebras and a lemma that will be useful in order to prove the equivalence between the variety of QNI-algebras and $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$, the equivalent quasivariety semantics corresponding to our calculus. In Section 3. we introduce the QNI-calculus and state the Deduction Theorem. In Section 4. we prove that QNI-logic is algebraizable and give a quasi-equational presentation for $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$. In Section 5. we prove that the class of QNI-algebras and $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$ coincide. We obtain as a corollary that the class $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$ has equationally definable principal congruences (EDPC). In Section 6. we give the equation that defines principal congruences on $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$, and we consider a few axiomatic extensions of QNI-logic, including (the negation-implication fragments of) intuitionistic logic and Nelson's constructive logic with strong negation.

2. Preliminaries

In this section we recall the definition and a few properties of the class of $\{\rightarrow, \sim\}$ -subreducts of quasi-Nelson algebras introduced in [Rivieccio, 2020a].

Given an algebra $\mathbf{A} = \langle A; \rightarrow, \sim, 0, 1 \rangle$ and elements $a, b \in A$, we will write $a \equiv b$ as a shorthand for $a \rightarrow b = b \rightarrow a = 1$. We will also employ the following abbreviations: $a \odot b := \sim(a \rightarrow \sim b)$ and

$$\beta(a, b, c) := (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow ((\sim a \rightarrow \sim b) \rightarrow ((\sim b \rightarrow \sim a) \rightarrow c))).$$

Definition 1 ([Rivieccio, 2020a], Definition 3.1). An algebra $\mathbf{A} = \langle A; \rightarrow, \sim, 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ is a *quasi-Nelson implication algebra* (QNI-algebra) if the following properties are satisfied, for all $a, b, c, d \in A$:

- (1) $1 \rightarrow a = a$
- (2) $a \rightarrow (b \rightarrow a) = a \rightarrow a = 0 \rightarrow a = 1$
- (3) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$
- (4) $\sim a \rightarrow (\sim b \rightarrow c) = (\sim a \odot \sim b) \rightarrow c$

$$(5) \beta(a, b, a) = \beta(a, b, b)$$

$$(6) \text{ if } \sim a \rightarrow \sim b = 1, \text{ then } \sim a \rightarrow (\sim a \odot \sim b) = 1$$

$$(7) a \odot (b \odot c) \equiv (a \odot b) \odot c$$

$$(8) a \odot b \equiv b \odot a$$

$$(9) \text{ if } a \equiv b \text{ and } c \equiv d, \text{ then } a \rightarrow c \equiv b \rightarrow d \text{ and } a \odot c \equiv b \odot d$$

$$(10) \sim a = \sim \sim \sim a$$

$$(11) \sim 1 = 0 \text{ and } \sim 0 = 1$$

$$(12) (a \rightarrow b) \rightarrow (\sim \sim a \rightarrow \sim \sim b) = 1.$$

$$(13) a \rightarrow \sim \sim a = 1$$

$$(14) \text{ if } a \rightarrow b = 1, \text{ then } a \odot c \rightarrow b \odot c = 1 \text{ and } c \odot a \rightarrow c \odot b = 1$$

$$(15) a \odot (a \rightarrow b) \equiv a \odot b$$

$$(16) a \odot b \equiv \sim \sim a \odot \sim \sim b$$

$$(17) \sim(a \rightarrow b) \equiv \sim(\sim \sim a \rightarrow \sim \sim b).$$

We shall denote by **QNI** the class of QNI-algebras. By definition, **QNI** is a quasivariety; it was shown in [Rivieccio, 2020a, Corollary 3.15] that **QNI** is in fact a variety.

Lemma 1 ([Rivieccio, 2020a], Lemma 3.3). *Let $\mathbf{A} \in \mathbf{QNI}$ and $a, b, c \in A$. Then:*

$$(1) (\sim a \odot \sim b) \rightarrow \sim a = (\sim a \odot \sim b) \rightarrow \sim b = 1.$$

$$(2) \text{ If } a \rightarrow b = 1 \text{ and } b \rightarrow c = 1, \text{ then } a \rightarrow c = 1.$$

$$(3) \text{ The relation } \leq \text{ defined by } a \leq b \text{ iff } (a \rightarrow b = 1 \text{ and } \sim b \rightarrow \sim a = 1) \text{ is a partial order on } A, \text{ with minimum } 0 \text{ and maximum } 1.$$

3. A Hilbert calculus for QNI logic

In this section we introduce a Hilbert-style calculus that determines a logic (in the sense of [Hamilton, 1978]) henceforth denoted by $\mathbf{L}_{\mathbf{QNI}}$. Our aim is to show that $\mathbf{L}_{\mathbf{QNI}}$ is regularly algebraizable, and that its equivalent algebraic semantics is precisely the variety **QNI**.

Fix a denumerable set **Atprop** of propositional variables. We use letters p, q, r etc. to refer to generic elements of **Atprop**. The propositional language $\mathcal{L}$ of $\mathbf{L}_{\text{QNI}}$ over **Atprop** is defined recursively as follows:

$$\varphi ::= p \mid \sim \varphi \mid \varphi \rightarrow \varphi.$$

Consistently with the above-introduced notation for QNI-algebras, we abbreviate $\varphi \odot \psi := \sim(\varphi \rightarrow \sim \psi)$. The Hilbert-calculus for $\mathbf{L}_{\text{QNI}}$ consists of the following axiom schemes:

$$\mathbf{AX1} \quad \varphi \rightarrow (\psi \rightarrow \varphi)$$

$$\mathbf{AX2} \quad (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$$

$$\mathbf{AX3} \quad \sim \sim \sim \varphi \rightarrow \sim \varphi$$

$$\mathbf{AX4} \quad (\varphi \rightarrow \psi) \rightarrow (\sim \sim \varphi \rightarrow \sim \sim \psi)$$

$$\mathbf{AX5} \quad \varphi \rightarrow \sim \sim \varphi$$

$$\mathbf{AX6} \quad (\varphi \odot (\varphi \rightarrow \psi)) \rightarrow (\varphi \odot \psi)$$

$$\mathbf{AX7} \quad \sim \sim \varphi \rightarrow (\sim \psi \rightarrow \sim(\varphi \rightarrow \psi))$$

$$\mathbf{AX8} \quad \sim(\varphi \rightarrow \psi) \rightarrow \sim \psi$$

$$\mathbf{AX9} \quad \sim(\varphi \rightarrow \psi) \rightarrow \sim \sim \varphi$$

$$\mathbf{AX10} \quad \sim(\varphi \rightarrow \varphi) \rightarrow \psi.$$

The only rule is *modus ponens* (MP): from φ and $\varphi \rightarrow \psi$, derive ψ .

The proof of the following result is the standard one by induction on the length of derivations.

Theorem 1 (Deduction-Detachment Theorem). *If $\Phi \cup \{\varphi\} \vdash_{\mathbf{L}_{\text{QNI}}} \psi$, then $\Phi \vdash_{\mathbf{L}_{\text{QNI}}} \varphi \rightarrow \psi$.*

Lemma 2.

$$(1) \quad \emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \varphi \rightarrow \varphi$$

$$(2) \quad \{\varphi \rightarrow \psi, \psi \rightarrow \chi\} \vdash_{\mathbf{L}_{\text{QNI}}} \varphi \rightarrow \chi.$$

4. $\mathbf{L}_{\text{QNI}}$ is (regularly) algebraizable

In this section we prove that $\mathbf{L}_{\text{QNI}}$ is regularly algebraizable. Using this result, we axiomatize the equivalent algebraic semantics of $\mathbf{L}_{\text{QNI}}$ via the algorithm of [Blok, Pigozzi, 1989, Theorem 2.17]. We then prove that the class of algebras thus obtained is equivalent to the class QNI given in Definition 1. Given the formula algebra $\mathbf{Fm}$, the associated set of equations, $Fm \times Fm$, will henceforth be denoted by Eq . We abbreviate an equation $\langle \varphi, \psi \rangle$ as $\varphi \approx \psi$.

Theorem 2. *A logic $\mathbf{L}$ is algebraizable if and only if there are a set of equations $E(\varphi) \subseteq Eq$ and a set of formulas $\Delta(\varphi, \psi) \subseteq Fm$, such that:*

(Alg) $\varphi \Vdash_{\mathbf{L}} \Delta(E(\varphi))$

(Ref) $\emptyset \vdash_{\mathbf{L}} \Delta(\varphi, \varphi)$

(MP) $\varphi, \Delta(\varphi, \psi) \vdash_{\mathbf{L}} \psi$

(Cong) for each n -ary operator $\bullet$,

$$\bigcup_{i=1}^n \Delta(\varphi_i, \psi_i) \vdash_{\mathbf{L}} \Delta(\bullet(\varphi_1, \dots, \varphi_n), \bullet(\psi_1, \dots, \psi_n)).$$

We call any such $E(\varphi)$ a set of defining equations and any such $\Delta(\varphi, \psi)$ a set of equivalence formulas of $\mathbf{L}$.

Definition 2. A logic $\mathbf{L}$ is regularly algebraizable when it is algebraizable and satisfies:

(G) $\varphi, \psi \vdash_{\mathbf{L}} \Delta(\varphi, \psi)$

for any non-empty set $\Delta(\varphi, \psi)$ of equivalence formulas.

Proposition 1. $\mathbf{L}_{\text{QNI}}$ is regularly algebraizable with $\Delta(\varphi, \psi) := \{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim\varphi \rightarrow \sim\psi, \sim\psi \rightarrow \sim\varphi\}$ and $E(\varphi) := \{\varphi \approx \varphi \rightarrow \varphi\}$.

Proof. As to (Alg), it suffices to prove that $\varphi \Vdash_{\mathbf{L}_{\text{QNI}}} \{\varphi \rightarrow (\varphi \rightarrow \varphi), (\varphi \rightarrow \varphi) \rightarrow \varphi, \sim\varphi \rightarrow \sim(\varphi \rightarrow \varphi), \sim(\varphi \rightarrow \varphi) \rightarrow \sim\varphi\}$. From right to left, thanks to Lemma 2.1, we have that $\varphi \rightarrow \varphi$ is a theorem and from $\varphi \rightarrow \varphi$ and $(\varphi \rightarrow \varphi) \rightarrow \varphi$ we get φ by using modus ponens. From left to right, we will prove that (i) $\varphi \vdash_{\mathbf{L}_{\text{QNI}}} \sim\varphi \rightarrow \sim(\varphi \rightarrow \varphi)$ and (ii) $\varphi \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\varphi \rightarrow \varphi) \rightarrow \sim\varphi$, the other two proofs easily follow from **AX1**. For (i),

1. φ	Assumption
2. $\varphi \rightarrow (\sim\varphi \rightarrow \sim(\varphi \rightarrow \varphi))$	AX7
3. $\sim\varphi \rightarrow \sim(\varphi \rightarrow \varphi)$	1, 2, MP

For (ii), notice that it is just **AX10**.

In order to prove **Ref**, it is necessary to show that $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \{\varphi \rightarrow \varphi, \sim \varphi \rightarrow \sim \varphi\}$, and it is Lemma 2.1. **(MP)** is a straightforward consequence of modus ponens.

As to **(Cong)**, we need to prove for each connective $\bullet \in \{\rightarrow, \sim\}$. For $(\sim)$, we need to prove that: (i) $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim \varphi \rightarrow \sim \psi$ and $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim \psi \rightarrow \sim \varphi$. The two deductions follow by our hypothesis. And we also need to prove (ii) $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim \sim \varphi \rightarrow \sim \sim \psi$ and $\{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim \sim \psi \rightarrow \sim \sim \varphi$. The two deductions follow from **AX4** together with our hypothesis. For $(\rightarrow)$, we need to prove that: (i) $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \vdash_{\mathbf{L}_{\text{QNI}}} (\varphi_1 \rightarrow \varphi_2) \rightarrow (\psi_1 \rightarrow \psi_2)$ and $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \vdash_{\mathbf{L}_{\text{QNI}}} (\psi_1 \rightarrow \psi_2) \rightarrow (\varphi_1 \rightarrow \varphi_2)$, they can be shown by Lemma 2.2; (ii) $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\varphi_1 \rightarrow \varphi_2) \rightarrow \sim(\psi_1 \rightarrow \psi_2)$ and $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\psi_1 \rightarrow \psi_2) \rightarrow \sim(\varphi_1 \rightarrow \varphi_2)$, we only prove the first one, the other proof is similar and hence omitted. Thanks to Theorem 1, in order to prove that $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\varphi_1 \rightarrow \varphi_2) \rightarrow \sim(\psi_1 \rightarrow \psi_2)$ is sufficient to prove that $\{\varphi_1 \rightarrow \psi_1, \psi_1 \rightarrow \varphi_1, \sim \varphi_1 \rightarrow \sim \psi_1, \sim \psi_1 \rightarrow \sim \varphi_1\} \cup \{\varphi_2 \rightarrow \psi_2, \psi_2 \rightarrow \varphi_2, \sim \varphi_2 \rightarrow \sim \psi_2, \sim \psi_2 \rightarrow \sim \varphi_2\} \cup \{\sim(\varphi_1 \rightarrow \varphi_2)\} \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\psi_1 \rightarrow \psi_2)$

1.	$\varphi_1 \rightarrow \psi_1$	Assumption
2.	$\sim \sim \varphi_1 \rightarrow \sim \sim \psi_1$	Lemma 3.6
3.	$\sim(\varphi_1 \rightarrow \varphi_2)$	Assumption
4.	$\sim \varphi_2$	Lemma 3.2
5.	$\sim \sim \varphi_1$	Lemma 3.3
6.	$\sim \varphi_2 \rightarrow \sim \psi_2$	Assumption
7.	$\sim \psi_2$	4, 6, MP
8.	$\sim \sim \psi_1$	2, 5, MP
9.	$\sim(\psi_1 \rightarrow \psi_2)$	7, 8, Lemma 3.1

It remains to prove **(G)**, that is, that $\varphi, \psi \vdash_{\mathbf{L}_{\text{QNI}}} \{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \psi \rightarrow \sim \varphi, \sim \varphi \rightarrow \sim \psi\}$. The deductions $\varphi, \psi \vdash_{\mathbf{L}_{\text{QNI}}} \varphi \rightarrow \psi$ and $\varphi, \psi \vdash_{\mathbf{L}_{\text{QNI}}} \psi \rightarrow \varphi$ follow from the deduction theorem. Now, that $\varphi, \psi \vdash_{\mathbf{L}_{\text{QNI}}} \sim \varphi \rightarrow \sim \psi$ and $\varphi, \psi \vdash_{\mathbf{L}_{\text{QNI}}} \sim \psi \rightarrow \sim \varphi$ follow from deduction theorem together with **AX7** and **AX10**. ■

We now apply [Czelakowski, Pigozzi, 2004, Theorem 30] to obtain a presentation of the equivalent algebraic semantics of $\mathbf{L}_{\text{QNI}}$.

Theorem 3. *Let $\mathbf{L}$ be a logic axiomatized by a set $\mathbf{Ax}$ of axioms and a set $\mathbf{Ru}$ of proper inference rules. Assume $\mathbf{L}$ is regularly algebraizable with a finite set of equivalence formulas $\Delta(\varphi, \psi) = \{\varepsilon_0(\varphi, \psi), \dots, \varepsilon_{n-1}(\varphi, \psi)\}$. Let $\top$ be a fixed but arbitrary theorem of $\mathbf{L}$. Then the unique equivalent quasivariety semantics of $\mathbf{L}$ is defined by the following identities and quasi-identities:*

- (1) $\varphi \approx \top$ for each $\varphi \in \mathbf{Ax}$.
- (2) $(\psi_0 \approx \top, \dots, \psi_p \approx \top)$ implies $\varphi \approx \top$, for each inference rule $\psi_0, \dots, \psi_p \vdash \varphi$ in $\mathbf{Ru}$.
- (3) $\Delta(\varphi, \psi) \approx \top$ implies $\varphi \approx \psi$.

In the above theorem, $\Delta(\varphi, \psi) \approx \top$ means that $\gamma \approx \top$ for each $\gamma \in \Delta(\varphi, \psi)$. Thanks to [Font, 2016, Proposition 3.47], we know that given a logic $\mathbf{L}$ that is regularly algebraizable with equivalent algebraic semantics the class $\mathbf{K}$, if φ is any theorem of $\mathbf{L}$, then φ is an algebraic constant of the class $\mathbf{K}$. Having this in mind, from now on we will let $1 := \varphi \rightarrow \varphi$ and $0 := \sim(\varphi \rightarrow \varphi)$. Applying Theorem 3 to our calculus $\mathbf{L}_{\text{QNI}}$, we obtain the following axiomatization of $\text{Alg}^*(\mathbf{L}_{\text{QNI}})$.

Proposition 2. The class $\text{Alg}^*(\mathbf{L}_{\text{QNI}})$ is axiomatized in the following way:

- (1) $\varphi \approx 1$ for each axiom φ of $\mathbf{L}_{\text{QNI}}$.
- (2) If $\varphi \approx 1$ and $\varphi \rightarrow \psi \approx 1$, then $\psi \approx 1$.
- (3) If $\varphi \rightarrow \psi \approx \psi \rightarrow \varphi \approx \sim \varphi \rightarrow \sim \psi \approx \sim \psi \rightarrow \sim \varphi \approx 1$, then $\varphi \approx \psi$.

5. $\text{Alg}^*(\mathbf{L}_{\text{QNI}}) = \text{QNI}$

In this section we prove that the class of QNI-algebras (Definition 1) coincides with the class $\text{Alg}^*(\mathbf{L}_{\text{QNI}})$ given in Proposition 2.

Proposition 3. $\text{Alg}^*(\mathbf{L}_{\text{QNI}}) \subseteq \text{QNI}$.

Proof. Given $\mathbf{A} \in \text{Alg}^*(\mathbf{L}_{\text{QNI}})$, we will prove that $\mathbf{A}$ satisfies all equations and quasi-equations given in Definition 1. Recall that from Proposition 2.3, we have that if $\varphi \rightarrow \psi \approx \psi \rightarrow \varphi \approx \sim \varphi \rightarrow \sim \psi \approx \sim \psi \rightarrow \sim \varphi \approx 1$, then $\varphi \approx \psi$.

The proofs of (1), (3), (4) and (5) are very similar to one another; we show the proof of (3) by way of an example. We have to prove that $(\varphi \rightarrow (\psi \rightarrow \gamma)) \approx ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$. We are going to show that $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow$

$((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \approx 1$, $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \gamma)) \approx 1$, $\sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \approx 1$ and $\sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow \sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \approx 1$. The desired result will then follow from Proposition 2.3. Thanks to Lemma 4.1 and Lemma 4.2, $\sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$, $\sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow \sim(\varphi \rightarrow (\psi \rightarrow \gamma))$ are theorems of $\mathbf{L}_{\text{QNI}}$ and therefore by Proposition 2.1 we conclude that $\sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) = 1$, $\sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow \sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \approx 1$, the same applies to $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$ and $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \gamma))$. As to (2), we have to prove that $\varphi \rightarrow (\psi \rightarrow \varphi) \approx 1$, $\varphi \rightarrow \varphi \approx 1$ and $0 \rightarrow \varphi \approx 1$, since $\varphi \rightarrow (\psi \rightarrow \varphi)$, $\varphi \rightarrow \varphi$ and $0 \rightarrow \varphi$ are theorems of $\mathbf{L}_{\text{QNI}}$ (**AX1**, Lemma 2.1 and **AX10**, respectively), thanks to Proposition 2.1 we have the desired equalities. As to (6), we have to prove that if $\sim\varphi \rightarrow \sim\psi \approx 1$, then $\sim\varphi \rightarrow (\sim\varphi \odot \sim\psi) \approx 1$. Since $(\sim\varphi \rightarrow \sim\psi) \rightarrow (\sim\varphi \rightarrow (\sim\varphi \odot \sim\psi))$ is a theorem of $\mathbf{L}_{\text{QNI}}$, Lemma 4.6, we conclude from Proposition 2.2 the equality. As to (7), it is just application of Lemmas 4.10 and 4.11. And for (8), notice that is just application of Lemma 4.3. As to (9), supposing that $\varphi \rightarrow \psi \approx 1$, $\psi \rightarrow \varphi \approx 1$, $\gamma \rightarrow \delta$, $\delta \rightarrow \gamma \approx 1$, we want to prove that $(\varphi \rightarrow \gamma) \rightarrow (\psi \rightarrow \delta) \approx 1$, $(\psi \rightarrow \delta) \rightarrow (\varphi \rightarrow \gamma) \approx 1$ and that $(\varphi \rightarrow \gamma) \odot (\psi \rightarrow \delta) \approx 1$, $(\psi \rightarrow \delta) \odot (\varphi \rightarrow \gamma) \approx 1$, since $(\psi \rightarrow \varphi) \rightarrow ((\gamma \rightarrow \delta) \rightarrow ((\varphi \rightarrow \gamma) \rightarrow (\psi \rightarrow \delta)))$ and $(\psi \rightarrow \varphi) \rightarrow ((\delta \rightarrow \gamma) \rightarrow ((\psi \rightarrow \delta) \rightarrow (\varphi \rightarrow \gamma)))$ are theorems of $\mathbf{L}_{\text{QNI}}$, Proposition 2.2 give us the desired equalities. The same idea is applied to $(\varphi \rightarrow \gamma) \odot (\psi \rightarrow \delta) \approx 1$, $(\psi \rightarrow \delta) \odot (\varphi \rightarrow \gamma) \approx 1$. In order to prove (10) notice that Lemma 3.4 and 3.5 give us that $\sim\varphi \rightarrow \sim\sim\sim\varphi$, $\sim\sim\sim\varphi \rightarrow \sim\varphi$, $\sim\sim\varphi \rightarrow \sim\sim\sim\sim\varphi$, $\sim\sim\sim\sim\varphi \rightarrow \sim\sim\varphi$ are theorems of $\mathbf{L}_{\text{QNI}}$ and therefore from Proposition 2.3, we conclude that $\sim\varphi \approx \sim\sim\sim\varphi$. In order to prove (11), notice that $0 := \sim(\varphi \rightarrow \varphi)$ and that $\sim\sim(\varphi \rightarrow \varphi)$ is a theorem of $\mathbf{L}_{\text{QNI}}$, then thanks to Proposition 2.1, $\sim\sim(\varphi \rightarrow \varphi) \approx 1$, i.e., $\sim 0 \approx 1$. As to (12), it is just application of **AX4**. From **AX5** and Proposition 2.1, we have that $\varphi \rightarrow \sim\sim\varphi \approx 1$, and it proves (13). As to (14), supposing that $\varphi \rightarrow \psi \approx 1$, we have to prove that $(\varphi \odot \gamma) \rightarrow (\psi \odot \gamma) \approx 1$ and that $(\gamma \odot \varphi) \rightarrow (\gamma \odot \psi) \approx 1$, and it is Lemma 4.3, Lemma 4.7 and Proposition 2.1. (15), (16) and (17) are **AX6**, Lemma 4.4 and Lemma 4.5 together with Proposition 2.2. ■

Proposition 4. $\text{QNI} \subseteq \text{Alg}^*(\mathbf{L}_{\text{QNI}})$.

Proof. First for all, we will prove that **A** satisfies $\varphi \approx 1$ for each axiom φ of $\mathbf{L}_{\text{QNI}}$. In the proof below, the notation $E(\mathbf{AX1})$ means $\varphi \approx 1$ being φ the first axiom of $\mathbf{L}_{\text{QNI}}$ and so on.

As to $E(\mathbf{AX1})$ and $E(\mathbf{AX2})$, notice that thanks to (3), $\varphi \rightarrow (\psi \rightarrow \varphi) \approx 1$ and $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \approx 1$. As to $E(\mathbf{AX3})$,

notice that thanks to (10) we have that $\sim\varphi \rightarrow \sim\sim\sim\varphi \approx 1$. As to $E(\mathbf{AX4})$, $E(\mathbf{AX5})$ and $E(\mathbf{AX6})$ notice that thanks to (12), (13) and (15), respectively, we have the equality. As to $E(\mathbf{AX7})$, notice that thanks to (4), we have that $\sim\sim\varphi \rightarrow (\sim\psi \rightarrow (\sim\sim\varphi \odot \sim\psi)) \approx (\sim\sim\varphi \odot \sim\psi) \rightarrow (\sim\sim\varphi \odot \sim\psi)$ and since $\varphi \rightarrow \varphi \approx 1$ from (2), we conclude that $\sim\sim\varphi \rightarrow (\sim\psi \rightarrow (\sim\sim\varphi \odot \sim\psi)) \approx 1$. Now, notice that $(\sim\sim\varphi \odot \sim\psi) \approx \sim(\sim\sim\varphi \rightarrow \sim\sim\psi)$ and thanks to (17), $\sim(\sim\sim\varphi \rightarrow \sim\sim\psi) \equiv \sim(\varphi \rightarrow \psi)$ and now from Lemma 1.2 we conclude that $\sim\sim\varphi \rightarrow (\sim\psi \rightarrow \sim(\varphi \rightarrow \psi)) \approx 1$. As to $E(\mathbf{AX8})$ and $E(\mathbf{AX9})$, notice that thanks to (16) and Lemma 1.1 we have that $\sim(\varphi \rightarrow \psi) \rightarrow \sim\sim\varphi \approx 1$ and $\sim(\varphi \rightarrow \psi) \rightarrow \sim\psi \approx 1$. As to $E(\mathbf{AX10})$, it is (2). In order to prove that If $\varphi \approx 1$ and $\varphi \rightarrow \psi \approx 1$, then $\psi \approx 1$, thanks to (1), since $\varphi \rightarrow \psi \approx \psi$, given that $\varphi \rightarrow \psi \approx 1$ we conclude that $\psi \approx \varphi \rightarrow \psi \approx 1$. It remains to prove that if $\varphi \rightarrow \psi \approx 1, \psi \rightarrow \varphi \approx 1, \sim\varphi \rightarrow \sim\psi \approx 1$ and $\sim\psi \rightarrow \sim\varphi \approx 1$, then $\varphi \approx \psi$. Thanks to Lemma 1.3 we conclude that from $\varphi \rightarrow \psi \approx 1$ and $\sim\psi \rightarrow \sim\varphi \approx 1$, $\varphi \leq \psi$ and from $\psi \rightarrow \varphi \approx 1$ and $\sim\varphi \rightarrow \sim\psi \approx 1$, $\psi \leq \varphi$. From the two inequalities we conclude that $\varphi \approx \psi$. ■

6. Extensions of $\mathbf{L_{QNI}}$ and $\mathbf{EDPC}$

In this section we look at some extensions of $\mathbf{L_{QNI}}$.

Proposition 5. The $\{\rightarrow, \sim\}$ -fragment of intuitionistic propositional logic is a strengthening of $\mathbf{L_{QNI}}$ obtained by adding any of the axioms below:

- (1) $(\varphi \rightarrow \psi) \rightarrow (\sim\psi \rightarrow \sim\varphi)$
- (2) $(\varphi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \sim\varphi)$
- (3) $(\varphi \rightarrow \sim(\varphi \rightarrow \varphi)) \rightarrow \sim\varphi$
- (4) $\sim(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \sim\varphi$
- (5) $\sim(\varphi \odot \psi) \rightarrow \sim(\psi \odot \varphi)$
- (6) $\sim(\sim\sim\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \sim((\varphi \rightarrow \varphi) \odot \sim\sim\varphi)$
- (7) $(\varphi \rightarrow \sim\varphi) \rightarrow \sim\varphi$
- (8) $\sim(\varphi \odot \varphi) \rightarrow \sim\varphi$.

Proof. Thanks to [Rivieccio, 2020a, Proposition 4.26 (iii)], we know that a QNI-algebra $\mathbf{A}$ is a bounded Hilbert algebra iff $\mathbf{A}$ satisfies any of the equations in [Rivieccio, 2020a, Lemma 4.25], the axioms (1), (2), (3), (4), (5) and (6) are equivalent to the equations (i), (ii), (iii), (v), (vi) and (vii), respectively.

In order to prove that (7) and (8) are equivalent, observe that $(\varphi \rightarrow \sim \varphi) \rightarrow \sim(\varphi \odot \varphi)$ and $\sim(\varphi \odot \varphi) \rightarrow (\varphi \rightarrow \sim \varphi)$ (See Lemma 4.9) are theorems of $\mathbf{L}_{\text{QNI}}$. In order to finish the equivalence, observe that $(\varphi \rightarrow \sim \varphi) \rightarrow (\varphi \rightarrow \sim(\varphi \rightarrow \varphi))$ and $(\varphi \rightarrow \sim(\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \sim \varphi)$ are theorems of $\mathbf{L}_{\text{QNI}}$. ■

In order to axiomatize the $\{\rightarrow, \sim\}$ -fragment of Nelson's constructive logic with strong negation, it is enough to add the involutive axiom $(\sim \sim \varphi \rightarrow \varphi)$ to $\mathbf{L}_{\text{QNI}}$. In order to obtain classical logic, we can add any of the following axioms to $\mathbf{L}_{\text{QNI}}$:

- (1) $\{(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \varphi, \sim(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \sim \varphi\}$
- (2) $(\sim \varphi \rightarrow \sim \psi) \rightarrow (\psi \rightarrow \varphi)$
- (3) $\{(\varphi \odot \varphi) \rightarrow \varphi, \sim(\varphi \odot \varphi) \rightarrow \sim \varphi\}$.

Proof. (1) and (2) are the logical counterparts identities in [Rivieccio, 2020a, Proposition 4.26 (iv)]. In order to prove that (1) and (3) are equivalent, suppose that $(\varphi \odot \varphi) \rightarrow \varphi$. Thanks to **AX6**, it follows that $(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow (\varphi \odot \varphi)$ and by using Lemma 2.2 it follows that $(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \varphi$. Now, suppose that $\sim(\varphi \odot \varphi) \rightarrow \sim \varphi$, as $\sim(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \sim(\varphi \odot \varphi)$ is a theorem of $\mathbf{L}_{\text{QNI}}$ (Lemma 4.8), it follows by Lemma 2.2 that $\sim(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \varphi$. ■

Remark 1. Consider the algebra $\mathbf{A}$ with universe $\{0, \frac{1}{2}, 1\}$ and operations given by the following tables:

$\sim$		$\rightarrow$	0	$\frac{1}{2}$	1
0	1	0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1
1	0	1	0	$\frac{1}{2}$	1

It is not difficult to show that $\mathbf{A}$ is a QNI-algebra that witnesses $\sim(\varphi \odot \psi) \not\models_{\mathbf{L}_{\text{QNI}}} \sim(\psi \odot \varphi)$. This entails that $\mathbf{L}_{\text{QNI}}$ is not self-extensional, see [Font, 2016, Definition 5.24] for a definition of self-extensionality.

It is well known that an algebraizable logic $\mathbf{L}$ with equivalent algebraic semantics a variety $\mathbf{K}$ has the Deduction-Detachment Theorem if and only if $\mathbf{K}$ has EDPC. Since $\mathbf{L}_{\text{QNI}}$ enjoys the DDT (Theorem 1), we know that $\mathbf{Alg}^*(\mathbf{L}_{\text{QNI}})$ has EDPC. We give below the equations witnessing this.

Definition 3 ([Blok, Pigozzi, 1994]). An algebra $\mathbf{A}$ has definable principal congruences (DPC) if there is a first-order formula $\varphi(x, y, z, w)$ such that, for all $a, b, c, d \in \mathbf{A}$,

$$\mathbf{A} \models \varphi[a, b, c, d] \iff c \equiv d \pmod{\Theta(a, b)}.$$

$\mathbf{A}$ has equationally definable principal congruences (EDPC) if φ can be taken to be a finite conjunction of identities $p_i(x, y, z, w) \approx q_i(x, y, z, w)$, for $i \in \omega$. A class $\mathbf{K}$ of algebras has EDPC if a single equation or conjunction of equations defines principal congruences on every member of $\mathbf{K}$.

Proposition 6 ([Font, 2016] Corollary 3.81). Let $\mathbf{L}$ be an algebraizable logic with equivalent algebraic semantics a variety $\mathbf{K}$. Then $\mathbf{L}$ satisfies the DDT if and only if $\mathbf{K}$ has EDPC.

Theorem 4 ([Rivieccio, Jansana, 2020] Corollary 34). *The term $\beta(x, y, z)$ satisfies:*

$$c \equiv d \pmod{\Theta(a, b)} \iff \beta(a, b, c) \approx \beta(a, b, d).$$

7. Conclusions and future work

As observed in [Rivieccio, 2020a], the translations that witness the algebraizability of (quasi-)Nelson logic can be defined using different choices of connectives. For the defining equation, one can let $E(\varphi) := \{\varphi \approx \varphi \rightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx 1\}$, or $E(\varphi) := \{\varphi \approx \varphi \rightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx \varphi \leftrightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx \varphi \Leftrightarrow \varphi\}$, where $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\varphi \Leftrightarrow \psi := (\varphi \rightarrow \psi) * (\psi \rightarrow \varphi)$, etc. For the equivalence formulas, one has for instance the following options: $\Delta(\varphi, \psi) := \{\varphi \Leftrightarrow \psi\}$, $\Delta(\varphi, \psi) := \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$, $\Delta(\varphi, \psi) := \{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim \varphi \rightarrow \sim \psi, \sim \psi \rightarrow \sim \varphi\}$, etc. Indeed, one can show that every fragment of quasi-Nelson logic containing either $\{\Rightarrow\}$ or $\{\Leftrightarrow\}$ or $\{\rightarrow, \sim\}$ is algebraizable.

With regards to future research, the above considerations suggest that a number of further fragments may be worthwhile looking at from the point of view of algebraizability. We note that, on the one hand, a successful characterization of a given fragment of quasi-Nelson logic may be easily specialized to the involutive case, therefore also yielding a characterization of the corresponding fragment of Nelson's constructive logic with strong negation¹. On the other hand, quasi-Nelson logic certainly has a greater number of non-equivalent fragments than Nelson's (and intuitionistic) logic, suggesting that the landscape may be quite complex.

While we know that the fragments containing either of the quasi-Nelson implications must be algebraizable, finding a complete axiomatization for them is a different question. Recent research experience on the topic suggests that the latter issue is related to the problem of giving a twist representation for the

¹As far as we know, the only studies on fragments of Nelson's logic are those by A. Monteiro's school [Monteiro, 1963; Brignole, Monteiro, 1967; Cignoli, 1986] on the Kleene algebra subreducts of Nelson algebras and Sendlewski's [Sendlewski, 1991].

corresponding classes of subreducts of quasi-Nelson algebras (see e.g. [Rivieccio, Spinks, 2021; Rivieccio, Jansana, 2020; Rivieccio, 2020a]). This endeavour, in turn, may require non-trivial adaptations of the algebraic constructions employed so far in the study of (quasi-)Nelson algebras. We mention below a few fragments regarding which a successful outcome may be anticipated, as well as some that seem less tractable.

If we enrich the $\{\rightarrow, \sim\}$ -fragment with the monoid or the lattice connectives of quasi-Nelson algebras, then we obtain logics/classes of algebras that appear to be well-behaved from the point of view that concerns us here. In particular, the $\{*, \rightarrow, \sim\}$ -fragment of quasi-Nelson logic (which can be shown to be equivalent to the $\{*, \Rightarrow, \sim\}$ -fragment) appears to be a particularly interesting one because of its connection with the theory of other residuated structures; this is currently the subject of ongoing research.

By contrast, the fragments that do not contain the negation ($\sim$) seem to lie beyond the applicability of the methods employed so far. The reason for this is too technical to be discussed in detail here, but it appears to be related to the observation that the property corresponding to the Nelson identity (not only as it appears in the present paper, but also in any of its alternative formulations: see [Rivieccio, Spinks, 2018; Rivieccio, Spinks, 2021]) can only be stated through the interaction of the negation with some other connective. In other words, what makes Nelson (and quasi-Nelson) algebras ‘Nelson’ seems to be precisely the interaction of the negation with some other connective (e.g. the interaction of the negation with the implications, of the negation with the lattice connectives, of the negation with the monoid conjunction, etc.).

Between the two extreme cases mentioned above, there also appear to be fragments that are not intractable in principle but (also for technical reasons) may prove to be hard to tackle. Of these we mention, as an example, the $\{\Rightarrow, \sim\}$ -fragment: this may turn out to be a particularly interesting case study, for it is certainly algebraizable and it contains the $\{\rightarrow, \sim\}$ -fragment, in the sense that the weak implication is definable. We leave this suggestion as a challenge for future investigations.

8. Appendix

The rules and valid formulas below are used in the proofs of Proposition 1 and Proposition 3.

Lemma 3.

$$(1) \sim\sim\psi, \sim\sim\varphi \vdash_{\mathbf{L}_{\text{QNI}}} \varphi \odot \psi$$

$$(2) \sim(\varphi \rightarrow \psi) \vdash_{\mathbf{L}_{\text{QNI}}} \sim\psi$$

- (3) $\sim(\varphi \rightarrow \psi) \vdash_{\mathbf{L}_{\text{QNI}}} \sim\sim\varphi$
 (4) $\varphi \vdash_{\mathbf{L}_{\text{QNI}}} \sim\sim\varphi$
 (5) $\sim\sim\sim\varphi \vdash_{\mathbf{L}_{\text{QNI}}} \sim\varphi$
 (6) $(\varphi \rightarrow \psi) \vdash_{\mathbf{L}_{\text{QNI}}} (\sim\sim\varphi \rightarrow \sim\sim\psi)$

Lemma 4.

- (1) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$
 (2) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \rightarrow \sim(\varphi \rightarrow (\psi \rightarrow \gamma))$
 (3) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$
 (4) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\varphi \odot \psi) \rightarrow (\varphi \odot (\varphi \rightarrow \psi))$
 (5) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\sim\sim\varphi \odot \sim\psi) \rightarrow \sim(\varphi \rightarrow \psi)$
 (6) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\sim\varphi \rightarrow \sim\psi) \rightarrow (\sim\varphi \rightarrow (\sim\varphi \odot \sim\psi))$
 (7) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\varphi \rightarrow \psi) \rightarrow (\varphi \odot \gamma) \rightarrow (\psi \odot \gamma)$
 (8) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \sim(\varphi \odot (\varphi \rightarrow \varphi)) \rightarrow \sim(\varphi \odot \varphi)$
 (9) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} \sim\sim(\varphi \rightarrow \sim\psi) \rightarrow (\varphi \rightarrow \sim\psi)$
 (10) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} (\varphi \odot (\psi \odot \gamma)) \rightarrow ((\varphi \odot \psi) \odot \gamma)$
 (11) $\emptyset \vdash_{\mathbf{L}_{\text{QNI}}} ((\varphi \odot \psi) \odot \gamma) \rightarrow (\varphi \odot (\psi \odot \gamma))$

<i>Proof.</i> [3.1]	1.	$\sim(\varphi \rightarrow (\psi \rightarrow \gamma))$	Assumption
	2.	$\sim\sim\varphi$	1, Lemma 3.3
	3.	$\sim(\psi \rightarrow \gamma)$	1, Lemma 3.2
	4.	$\sim\sim\psi$	3, Lemma 3.3
	5.	$\sim\gamma$	3, Lemma 3.2
	6.	$\psi \rightarrow (\varphi \rightarrow \psi)$	AX1
	7.	$\sim\sim\psi \rightarrow \sim\sim(\varphi \rightarrow \psi)$	6, Lemma 3.6
	8.	$\sim\sim(\varphi \rightarrow \psi)$	4, 7, MP
	9.	$\sim(\varphi \rightarrow \gamma)$	2, 5, AX7
	10.	$\sim((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$	8, 9, AX7
			■

<i>Proof.</i> [3.3]	1.	$\varphi \odot \psi$	Assumption
	2.	$\sim\sim\varphi$	1, Lemma 3.3
	3.	$\sim\sim\psi$	2, Lemma 3.2
	4.	$\psi \odot \varphi$	2, 3, Lemma 3.1
			■

Proof. [3.4]	1.	$\varphi \odot \psi$	Assumption	■
	2.	$\sim \sim \varphi$	1, Lemma 3.3	
	3.	$\sim \sim \psi$	2, Lemma 3.2	
	4.	$\psi \rightarrow (\varphi \rightarrow \psi)$	AX1	
	5.	$\sim \sim \psi \rightarrow \sim \sim (\varphi \rightarrow \psi)$	AX4	
	6.	$\sim \sim (\varphi \rightarrow \psi)$	3, 5, MP	
	7.	$\varphi \odot (\varphi \rightarrow \psi)$	2, 6, Lemma 3.1	

Proof. [3.5]	1.	$\sim \sim \varphi \odot \sim \psi$	Assumption	■
	2.	$\sim \sim \sim \sim \varphi$	1, Lemma 3.3	
	3.	$\sim \sim \sim \psi$	2, Lemma 3.2	
	4.	$\sim \sim \varphi$	3, Lemma 3.4	
	5.	$\sim \psi$	4, Lemma 3.5	
	6.	$\sim \sim \varphi \rightarrow (\sim \psi \rightarrow \sim (\varphi \rightarrow \psi))$	AX7	
	7.	$\sim \psi \rightarrow \sim (\varphi \rightarrow \psi)$	4, 6, MP	
	8.	$\sim (\varphi \rightarrow \psi)$	5, 7, MP	

Proof. [3.6]	1.	$\sim \varphi \rightarrow \sim \psi$	Assumption	■
	2.	$\sim \varphi$	Assumption	
	3.	$\sim \psi$	1, 2, MP	
	4.	$\sim \sim \sim \varphi$	2, Lemma 3.5	
	5.	$\sim \sim \sim \psi$	3, Lemma 3.5	
	6.	$\sim \sim \sim \varphi \rightarrow (\sim \sim \sim \psi \rightarrow \sim (\sim \varphi \rightarrow \sim \sim \psi))$	AX7	
	7.	$\sim \sim \sim \psi \rightarrow \sim (\sim \varphi \rightarrow \sim \sim \psi)$	4, 6, MP	
	8.	$\sim (\sim \varphi \rightarrow \sim \sim \psi)$	5, 7, MP	

Proof. [3.7]	1.	$\varphi \rightarrow \psi$	Assumption	■
	2.	$\varphi \odot \gamma$	Assumption	
	3.	$\sim \sim \varphi$	2, Lemma 3.3	
	4.	$\sim \sim \gamma$	2, Lemma 3.2	
	5.	$(\varphi \rightarrow \psi) \rightarrow (\sim \sim \varphi \rightarrow \sim \sim \psi)$	AX4	
	6.	$\sim \sim \varphi \rightarrow \sim \sim \psi$	1, 5, MP	
	7.	$\sim \sim \psi$	3, 6, MP	
	8.	$\psi \odot \gamma$	4, 7, Lemma 3.1	

Proof. [3.10]	1.	$(\varphi \odot (\psi \odot \gamma))$	Assumption	
	2.	$\varphi \odot \sim(\psi \rightarrow \sim\gamma)$	Definition	
	3.	$\sim(\varphi \rightarrow \sim\sim(\psi \rightarrow \sim\gamma))$	Definition	
	4.	$\sim\sim\varphi$	3, Lemma 3.3	
	5.	$\sim\sim\sim(\psi \rightarrow \sim\gamma)$	4, Lemma 3.2	
	6.	$\sim(\psi \rightarrow \sim\gamma)$	5, Lemma 3.5	■
	7.	$\sim\sim\psi$	6, Lemma 3.3	
	8.	$\sim\sim\gamma$	7, Lemma 3.2	
	9.	$(\varphi \odot \psi)$	4, 7, Lemma 3.1	
	10.	$\sim\sim(\varphi \odot \psi)$	9, Lemma 3.4	
	11.	$((\varphi \odot \psi) \odot \gamma)$	8, 10, Lemma 3.1	

Acknowledgements. T. Nascimento was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nivel Superior – Brasil (CAPES) – Finance Code 001. U. Rivieccio acknowledges partial funding by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil), under the grant 313643/2017-2 (Bolsas de Produtividade em Pesquisa – PQ). Special thanks are due to Ramon Jansana, for several useful comments on earlier versions of the paper.

References

- Blok, Pigozzi, 1989 – Blok, W.J., Pigozzi, D. “Algebraizable Logics”, *Memoirs of the American Mathematical Society*, 1989, Vol. 77.
- Blok, Pigozzi, 1994 – Blok, W.J., Pigozzi, D. “On the structure of varieties with equationally definable principal congruences III”, *Algebra Universalis*, 1994, Vol. 32, pp. 545–608.
- Brignole, Monteiro, 1967 – Brignole, D., Monteiro, A. “Caracterisation des algèbres de Nelson par des égalités. I, II”, *Proceedings of the Japan Academy*, 1964, Vol. 43, pp. 279–285.
- Cignoli, 1986 – Cignoli, R. “The class of Kleene algebras satisfying an interpolation property and Nelson algebras”, *Algebra Universalis*, 1986, Vol. 23, pp. 262–292.
- Czelakowski, Pigozzi, 2004 – Czelakowski, J., Pigozzi, D. “Fregean logics”, *Annals of Pure and Applied Logic*, 2004, Vol. 127, pp. 17–76.
- Font, 2016 – Font, J.M. “Abstract Algebraic Logic. An Introductory Textbook”, *College Publications*, 2016.
- Galatos et al., 2007 – Galatos, N., Jipsen, P., Kowalski, T., and Ono, H. “Residuated Lattices: an algebraic glimpse at substructural logics”, *Studies in Logic and the Foundations of Mathematics*, 2007.
- Hamilton, 1978 – Hamilton, A.G. “Logic for Mathematicians”, *Cambridge University Press*, 1978.
- Liang, Nascimento, 2019 – Liang, F., Nascimento, T. “Algebraic Semantics for quasi-Nelson Logic”, *Lecture Notes in Computer Science*, 2019, Vol. 11541, pp. 450–466.

- Monteiro, 1963 – Monteiro, A. “Construction des algèbres de Nelson finies”, *Bulletin of the Polish Academy of Sciences*, 1963, Vol. 11, pp. 359–362.
- Nelson, 1949 – Nelson, D. “Constructible falsity”, *The Journal of Symbolic Logic*, 1949, Vol. 14, pp. 16–26.
- Rivieccio, 2020a – Rivieccio, U. “Fragments of quasi-Nelson: The Algebraizable core”, *submitted*.
- Rivieccio, 2020b – Rivieccio, U. “Representation of De Morgan and (Semi-)Kleene lattices”, *Soft Computing*, 2020, Vol. 24 (12), pp. 8685–8716.
- Rivieccio, 2020c – Rivieccio, U. “Fragments of quasi-Nelson: two negations”, *Journal of Applied Logic*, 2020, Vol. 7, No. 4, pp. 499–559.
- Rivieccio, Jansana, 2020 – Rivieccio, U., Jansana, R. “Quasi-Nelson algebras and fragments”, *Mathematical Structures in Computer Science*, pp. 1–29. DOI: 10.1017/S0960129521000049.
- Rivieccio, Spinks, 2018 – Rivieccio, U., Spinks, M. “Quasi-Nelson algebras”, *Electronic Notes in Theoretical Computer Science*, 2019, Vol. 344, pp. 169–188.
- Rivieccio, Spinks, 2021 – Rivieccio, U., Spinks, M. “Quasi-Nelson; or, non-involutive Nelson algebras.”, *Algebraic Perspectives on Substructural Logics*, ed. by D. Fazio, A. Ledda, and F. Paoli, Cham: Springer 2021, pp. 133–168.
- Sendlewski, 1991 – Sendlewski, A. “Topologicality of Kleene algebras with a weak pseudocomplementation over distributive p-algebras”, *Reports on Mathematical Logic*, 1991, Vol. 25, pp. 13–56.