OPEN RUSSIAN-FINNISH COLLOQUIUM

ORFIC

ON LOGIC

MMXII

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The present volume is devoted to the 'Open Russian-Finnish Colloquium on Logic' (ORFIC), held at the Saint-Petersburg State University, on June 14–16, 2012. Among the participants there were such prominent Finnish logicians as Jaakko Hintikka, Ilkka Niiniluoto and Gabriel Sandu. The volume covers the most interesting results recently obtained in different areas of research in logic.

The volume is of interest to everyone, concerned in modern logic.

Photo of J. Hintikka by Veikko Somerpuro

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Данный сборник поясвящен «Открытому Российско-Финскому коллоквиуму по логике» (ОРФиК), который состоялся в Санкт-Петербургском государственном университете 14–16 июня 2012 г. Среди участников были такие выдающиеся финские логики как Я. Хинтикка, И. Ниинилуото и Г. Санду. В сборник включены наиболее интересные результаты, полученные в различных областях логики за последнее время.

Сборник предназначен для всех интересующихся современной логикой.

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Preface

(The history of Finnish-Soviet Logic Colloquium)[1]

Finnish-Soviet Logic Colloquium has a long and interesting history. 24 May 1971 a Protocol on cooperation between Academy of Sciences of the USSR and Academy of Finland was concluded. Then 15 May 1980 an Agreement on scientific cooperation between Academy of Sciences of the USSR and Academy of Finland was concluded. At last 25 May 1993 an Agreement on scientific cooperation between Russian Academy of Sciences and Academy of Finland was concluded. On the basis of these documents nine Finnish-Soviet[Russian] Logic Colloquiums were arranged.

Somewhere around 1975 J. Hintikka[2] and V.A. Smirnov[3] have agreed to hold Finnish-Soviet Conference on logic. The coopera-

[1]Beginning with the 7th Colloquium — 'Finnish-Russian Logic Colloquium'

[2]Kaarlo Jaakko Juhani Hintikka (born January 12, 1929) — Finnish logician and philosopher. He is regarded as the founder of formal epistemic logic, model set, and of game semantic for logic. He is known also as one of the architects of distributive normal forms, possible-worlds semantics, and tree methods. In recent decades, he has worked mainly on game semantics and on independence-friendly (IF) logic known for its 'branching quantifiers' which he believes do better justice to our intuitions about quantifiers than does conventional first-order logic (see the paper of Hintikka J. and G. Sandu *A revolution in logic?* in 'Nordic Journal of Philosophical Logic' 1(2):169–183, 1996). Note that IF-logic has caused the big interest in a logical world. He also has done important exegetical work on Aristotle, Kant, Wittgenstein, and C.S. Peirce. In 1998 Hintikka wrote *The Principles of Mathematics Revisited* which takes an exploratory stance comparable to that Russell made with his *The Principles of Mathematics* in 1903. A comprehensive examination of his thought appeared in 2006 as the volume *The Philosophy of Jaakko Hintikka* in the series Library of Living Philosophers.

[3]Vladimir Alexandrovich Smirnov (2 March 1931 – 12 February 1996) — Russian logician and philosopher. Undoubtedly, the most important logical

tion started with the **First Finnish-Soviet Logic Colloquium** at Jyväskylä, Finland, on June 29 – July 6, 1976. The proceedings of that Colloquium have been published under the title *Essays in Mathematical and Philosophical Logic*, edited by J. Hintikka, I. Niiniluoto, and E. Saarinen (D. Reidel Publishing Company, Dordrecht, 1979)[4].

The Second Finnish-Soviet Logic Colloquium was held in Moscow, at the Institute of Philosophy, on December 3–7, 1979. The Finnish delegation (11 scientists) led G. von Wright[5] — the teacher of J. Hintikka. The Conference generated tremendous interest in the Soviet Union and was attended by over 200 people from different logical centers of the country (Novosibirsk, Tbilisi, Kiev, Kalinin, Minsk, Baltics etc.) By the beginning of the Conference four collections of Abstracts were published in English: *Modal and Tense Logics*, Moscow, 1979; *Relevant Logic and the Theory of Inference*, Moscow, 1979; *Logical Analysis of Natural Languages*,

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work by Smirnov is his doctoral thesis *Formal Deduction and Logical Calculi* (1972). This book (in Russian) has become a classic. It contained a number of important technical results in the field of modern formal logic and was full of new ideas. For the first time ever in the world literature the study of logics without contraction rule was begun and the decision problem for such logics was examined. In the very same book the problem of the classification of logical calculi was formulated and discussed for the first time. Also he obtained results in relevant logics, definability and logical relations between theories, modal-temporal logics, combined logics, syllogistics, Lesniewski's systems, completely free logics, and proof theory. A comprehensive examination of his thought see in special issue in memory of V.A. Smirnov edited by K. Segerberg (*Studia Logica* 66(2), 2000) with the introductory and survey papers by A.S. Karpenko.

[4]Also *Proceedings of the fourth Scandinavian logic symposium* are included in this volume.

[5]Georg Henrik von Wright (14 June 1916 – 16 June 2003) was one of the most prominent European philosophers of the 20^{th} century, who succeeded Ludwig Wittgenstein as professor at the University of Cambridge. Von Wright's work included important writings on philosophical logic, philosophy of science, philosophy of language, philosophy of mind and ethic. His 1951 books, *An Essay in Modal Logic* and *Deontic Logic*, were landmarks in the postwar rise of formal modal logic and its deontic version. For philosophy of science see two of his most famous books, *Explanation and Understanding* (1971), and *Freedom and Determination* (1980). Beyond logic, analytic philosophy and politics, he wrote on classical Russian literature and a variety of other subjects. From 1968 to 1977, he was chancellor of Abo Academy in Finland.

Moscow, 1979; *Reports of Finnish participants*, Moscow, 1979. Reports of Soviet and Finnish participants in the Colloquium were published in English under the title *Intensional Logic: Theory and Application*, in 'Acta Philoshica Fennica' 35, 1982 (ed. by I. Niiniluoto and E. Saarinen) and in English under the title *Modal and Intensional Logic and their Application to Problem of Methodology of Sciences*, ed. by V.A. Smirnov, A.S. Karpenko and E.A. Sidorenko, NAUKA Publishers, Moscow, 1984.[6]

Due to the extraordinary success of the Colloquium held, the decision was made to hold such colloquiums every two years. Although, V.A. Smirnov has proposed to invite polish logicians to the three-sided cooperation continuing the two earlier bilateral Finnish-Soviet Logic Colloquium. By this time an Agreement on scientific cooperation between Academy of Sciences of the USSR and Polish Academy of Sciences was also concluded[7]. The first Finish-Polish-Soviet Logic Conference was held at Polanica Zdrój, Poland, on September 7–12, 1981. Proceeding of this Conference was published in special issue 'Studia Logica' XLII(2/3), 1983 (ed. by I. Niiniluoto and J. Zygmunt). Unfortunately, at this point three-sided cooperation was ended.

The Third Finnish-Soviet Logic Colloquium was held at Helsinki, on May 23–27, 1983. The Colloquium was also attended by Swedish logicians. Reports of the participants in the Colloquium were published in English in a special issue of the journal 'Synthese' 61(1), 1986 (Guest Editor I. Niiniluoto)[8].

The Fourth Finnish-Soviet Logic Colloquium was held at Telavi (Georgia), on May 20–24, 1985. The Georgian side, as usual, has amazed everyone by its hospitality and friendliness. By the beginning of the Conference, Abstracts in Russian have been published: *Intensional Logics and Logical Structure of Theories*, (eds. V.A. Smirnov and M.N. Bezhanishvili), Mecniereba, Tbilisi, 1985.

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[6] Also *Proceedings of the Soviet-Hungarian logic symposium on modal logics* are included in this volume.

[7] Let us note that three bilateral Polish-Soviet Logical Conferences were held in total.

[8] A welcome reception for all the participants was held by G.H. von Wright at his house.

Part of the Colloquium, concerning to problems of provability logic, syllogistics, and logical structure of scientific theories was published in Russian under the title *Intensional Logics and Logical Structure of Theories*, (eds. V.A. Smirnov and M.N. Bezhanishvili), Mecniereba, Tbilisi, 1988. The other part was published also in Russian under the title *Investigations in Non-Classical Logics* (editor in chief V.A. Smirnov), NAUKA Publishers, Moscow, 1989.

The Fifth Finnish-Soviet Logic Colloquium was held at Helsinki, on May 26–30, 1987. Note, that Congress of Logic, Philosophy and Methodology of Science was held at Moscow in the same year on August.

The Sixth Finnish-Soviet Logic Colloquium was held at Moscow, on June 10–16, 1989. At this point the Finnish delegation was joined by Gabriel Sandu, who had emigrated from Soviet Romania to Finland, his participation undoubtedly livened up the work of the conference. Thus, Finnish-Soviet Logic Colloquium started to gradually assume more open nature.

Between Sixth and Seventh Colloquiums a grandiose event in the world of logic took place. The 9^{th} International Congress of Logic, Methodology and Philosophy of Science was held at Uppsala, Sweden, on August 7–14, 1991. Part of Soviet delegation made its way to Sweden via Helsinki by steamship, where a meeting with Jaakko Hintikka took place during which he delivered an interesting lecture.

The Seventh Finnish-Russian Logic Colloquium was held at Helsinki — Turku — Lahti, on May, 1992.

Between the seventh and eighth Colloquium there were two events. My old friends Ilkka Niiniluoto and Veikko Rantala took part in International Conference 'Philosophy of Science' which was held at Moscow, on February, 1993. The second event was very important for me. I was invited by I. Niiniluoto to visit the University of Helsinki on April 28 – June 28, 1994. During the work there the great aid was also granted by G. Sandu.

The Eighth Finnish-Russian Logic Colloquium was held at Moscow, on June, 1995.

As late as after V.A. Smirnov's death I. Niiniluoto[9] has organized regular **The Ninth Finnish-Russian Logic Colloquium** which was held at Helsinki, on October 22–28, 1997. As usual, 10 people were invited. But only three participants made it to the Colloquium. At that moment the history of Finnish-Russian Logic Colloquiums stopped for 15 long years.

In 2011 on January 24 – February 4, The Faculty of Philosophy of St. Petersburg University was holding the Second Winter School 'Bridge to Logic', participated in by J. Hintikka, I. Niiniluoto, and G. Sandu. Here G. Sandu (the Chairman of the Department of Philosophy in the University of Helsinki) and I. Mikirtumov (the Chairman of the Department of Logic, Faculty of Philosophy of St. Petersburg University) agreed to hold a Conference on logic. After a conversation with Mikirtumov, I remembered glorious Finnish-Russian Logic Colloquium and told him about it. I agreed to publish all plenary reports in 'Logical Investigations'. To mark the novelty of the conference Mikirtumov added the word 'Open' to the title. As a result **Open Russian-Finnish Colloquium on Logic** (***ORFIC***) was held at St. Petersburg University, on June 14–16, 2012. By the beginning of the Conference, Abstracts (except for plenary reports) in Russian have been published: *Logic, Language and Formal Models. ORFIC*, (eds. Y. Chernoskutov, E. Lisanyuk, and I. Mikirtumov), St. Petersburg University, 2012.

Present special issue of 'Logical Investigations' includes plenary as well as section reports delivered at ***ORFIC***. Besides, in *ADDITION* section we have included the authors who eagerly wanted to participate in ***ORFIC*** but were unable due to varying reasons.

Alexander S. Karpenko

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[9]Rector of the University of Helsinki 2003–2008, chancellor of the University of Helsinki, beginning 1 June 2008.

Logic and object theory in 19th century: from Bolzano to Frege

Yury Yu. Chernoskutov[1]

ABSTRACT. The milestones of the object theory formation in the course of 19th century discussions in philosophy of logic are considered. The view, that the process mentioned was typical first of all for the Austrian tradition in logic and philosophy, is exposed. The hypothesis of the possible impact of that kind of approaches on the development of Frege's logical ideas is examined.

Keywords: object theory, content and object of concept, school of Brentano, Frege

1 Introduction

In the paper we are going to trace the formation of the object theory and its connection with the development of logic in 19$^{\text{th}}$ century. There were three views on the stuff and subject of logic in that century. The first prevailed on the continent, its general tenets were found by I. Kant. The (pure) Logic was considered here as a science of the form of thought. The principles of second were laid down by R. Whately, and in fact that principles formed the paradigm of British tradition in the philosophy of logic. The logic was considered here as an inquiry of reasoning, and its formal character was explained by the fact that it deals with the form of language expression. The third approach, which I'd want to designate as 'objectual', has been developed mainly in the framework of the Austrian tradition of logic and philosophy. According to this view, logic is a theory of science (Wissenschaftslehre); logic may be called formal, because it deals with the form of object in general. The Austrian tradition was not so influential and wide-known as both former. Moreover,

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up to now it has been rarely identified as independent tradition. I hope that filling that gap will provide more adequate look on the process of modern logic formation.

Kant's underestimate of the possibilities of *pure logic* is well-known. According to him, it examines forms of understanding a priori, i.e. of the cognitive faculty, which does not have direct connection to any object of knowledge and consequently, being viewed in itself, is contentless. It is only transcendental logic which provides us with the capacity to deal with the 'object in general'. Afterwards the arguments of Kant were reinforced by Adolf Trendelenburg, due to whom, to the word, the term *pure logic* was changed by the habitual nowadays label *formal logic.* This had led to the situation that in the course of 19^{th} century proper *formal logic* did not inspire German philosophers: one should hardly point learnbook or monograph titled as *Formale Logik* and authored by German writer.

2 J.F. Herbart

The first important move to the objectual interpretation of the subject of logic was made by German philosopher Johann Friedrich Herbart (1776–1841) and his school. He did not made object the subject-matter of logical inquiry; his move consisted in shifting that subject-matter from the sphere of epistemology or psychology in the direction of ontology. He is wide-known due to the 'Herbartianism' in pedagogy, history of psychology acknowledge his merits as one of pioneers in mathematization of psychology; but in the histories of philosophy he attracts very few attention nowadays. The situation contrasts sharply to the role his ideas played in the progress of philosophy during his life-time. Herbart was one of the first influential antagonists of German idealism and for a long time his school was in fact the only force advocating rational spirit in philosophy on the background of nearly exclusive domination of speculative constructions. He defined philosophy as 'reworking [bearbeitung] of concepts'. Accordingly, different sections of philosophy supervise different stages, or kinds of that reworking. The first section, logics, is to make concepts clear (draw sharp borderlines between diverse concepts) and distinct (strict distinguish the features of certain concept from each other). The second, metaphysics, deals with modification of concepts. Finally, esthetics (which includes ethics) accounts for valuating of concepts. Thus, *concept* serves as a central object of not only logic, but also of Herbartian philosophy in general.

Herbart considers concepts as kind of ultimate entities, analysis of which in logic excludes any questions concerning their genesis. Logic deals with concepts as something pre-given, ready-made; it should not ask where they come from. His *Hauptpunkte der Logik* (1808) starts with a claim: 'Logic deals with representations. But it does not deal with the act of representing: thus neither with the way and manner by means of which we arrive at them, nor with the mental states [Gemutszustande] to which we are moved by this'. It is concerned only with this, '*what is represented*'. That represented turns to the subject of logic insofar as it is 'afore grasped, singled out, conceived. This is why it is named a concept'. [6, S. 467]. In his next work, *Lehrbuch zur Einleitung in die Philosophie* (1813) he unfold his views in more details. The concept is described here as 'thoughts, considered in view of what is thought through them', and the latter is explicitly contraposed to the idea of concept as process or 'activity of our mind' [6, S. 77]. Thus, Herbart sharply distinguishes representation as (psychical) act of mind from represented, thinking from thinkable. The distinction provides preconditions for shifting the subject of logic from the scope of epistemology or psychology to that of ontology. The fact partially explains the ease of later reception of Herbartianism by the Austrian logicians and philosophers. Herbart did not introduce into logic the category of 'object' (whether actual or abstract), but, in view of his doctrine of 'what is represented' and of 'thinkable', the step to object looks quite natural and coherent. We should point that in the framework of Kantianism the step is hardly possible.

It is hard to keep oneself from the comparison of Herbart's 'representable' with Bolzanian 'representation in itself'. The former does not describe his concepts as 'objective' as the latter did on his essences. Herbart didn't go beyond negative characteristics and informed us just of what concepts are not. Namely, they are 'neither real objects, nor actual [wirkliche] acts of thinking' [6, S. 78]. In other words, concepts belong to some intermediate domain between external things and internal psychical acts. He escapes from positive characterizing the nature of concepts as well as of this intermediate sphere, but we can suppose that he keeps in mind some kind of platonistic world. Actually, each concept 'is given as if in a single exemplar'; what is to the question of relation between concepts and thinking, Herbart says that 'thinking of one and the same concept may be reproduced over and over again' in the consciousness of different human beings, but the fact 'does not bring the duplication of concept' [6, S. 78].

So, the nature of concepts is completely irrelevant to the properties of our cognitive faculties, they are not product of the mind activity. As consequence, Herbart expels any considerations of thinking beyond the competence of logic. Thinking is 'just a mediator, a kind of cart which brings concepts into one place' [6, S. 91]. It is easy to see that logic studies not properties of 'cart', but properties of what is shipped by the 'cart'. When two concepts meet each other in the process of thinking, they are 'suspended and form a question' [ibid.]. Making an answer to that question, we commit a judgment. This interpretation of judgment is very close to doctrines of Brentano and Frege, for both of them explained judgment as affirming (in case of Brentano also rejecting) of content represented. But Herbart believed that any kind of valuating is a psychological, or at least extra-logical process and, consequently, we should evade considerations of that sort as far as we are inside logic. Hence except this, psychological sense of judgment, he adds the logical sense — combination of subject and predicate. Due to distinguishing the judgment in logical sense from judgment as evaluating act, Herbart rejected the Kantian view that different acts of thinking generate different sorts of judgments. For him, the difference of categorical, hypothetical and disjunctive judgments is not the difference in the logical sense; it 'belongs completely to the language form' [6, S. 473]

Summing up, we should conclude that Herbart has distinguished some special sphere, for which he managed to propose only 'apophatic' description, which takes intermediate place between thinking and actual world, and which constitutes the proper subject of logic.

3 B. Bolzano

The next, more decisive move in the direction of object theory and ontological exposition of logic was made by Bernard Bolzano (1781–1848). His early interest was aimed at problems of foundations of geometry, namely he tried to examine the independency of Euclidian fifth postulate. In course of this work he came to belief that Kantian views on the nature of logics and mathematics are wrong. In particular, he discarded the belief that intuition lays in the ground of all mathematics. Another crucial conclusion was that mathematics needs more rigorous logical tools for carrying out its proofs. But formal logic available at the period could not serve the aim satisfactorily, and Bolzano, step by step, started developing his own system of logic. Those efforts resulted in extensive and grandiose, in four volumes treatise *Theory of science* [Wissenschaftslehre] (1837). In the introduction he proposes to take 'sentences in itself [satz an sich], representations in itself [vorstellung an sich] and truths in itself [wahrheit an sich] as a proper subject-matter of logic' [5, Bd. 1, S. 63]. Under those essences in itself Bolzano means objective content of sentences and representations, which is independent of the way of its expressing, of the way of its thinking, of our attitudes, at last of the very fact whether we think of it or not. He separates distinctly the representation in itself from the thinkable, subjective representation: the former is in no way generated by the latter, neither is any special kind of it. He rather prefer to make the latter in some way subordinate to the former, when he says, that 'objective representation… might be named the matter of subjective representation' [4, p. 277, § 271]. The principal property of sentences in itself and representations in itself, which differs them from thinkable sentences and representations, as we can see, consists in their objectivity. Logic, according to Bolzano, is a formal science, but it is due to the fact that it considers the *forms of 'propositions-in-itself'*, not the *forms of thought*. Thereby logics may not be viewed as objectless knowledge and qualifying it as formal will not serve as verdict in unproductiveness.

Introducing of ideal entities into logic is not the only novelty. Along with objective representation in itself Bolzano distinguishes object [Gegenstand] of representation: 'Under the object of representation I mean that (existing or non-existing something), of which it is said that it is represented, or that there is a representation of it' [5, Bd. 1, § 49]. Bolzano repeatedly stresses that object of representation is an independent entity, which should not be mixed with the representation in itself: 'one should distinct sharply representation in itself and object of representation'; it should not be 'confused with the object of representation' [4, p. 277, § 271]. The object of representation plays an essential role in his logic: most of principal logical relations are defined in terms of object. Moreover, he in fact excludes from the consideration sentences, which do not deal with any object. He believes that 'if not all sentences, than at least all true sentences are to have an object they deal with' [4, p. 208, § 196].

The logical innovations designed by Bolzano were essentially intended to make logical apparatus applicable to the mathematical reasoning, and more generally, to make it appropriate as a theory of science, *Wissenschaftslehre*. Those innovations were discussed more than once, and I'd want just to pay attention that above described shifting to ontology was an important component of implementing this intention. The fact is that Bolzano detaches acutely the subject of logic from epistemology and psychology, and makes it to inquire the formal properties and relations of objective entities. Besides, Bolzano first introduced into logic three-partial structure '(objective) representation in itself — (subjective) thinkable representation — object of representation'. Subsequently we'll face repeatedly the structure in the doctrines of Austrian philosophers, in various clothes and not only in logic.

4 R. Zimmerman

Robert Zimmerman (1824–1898) was one of last pupils of Bolzano. Since 1852 he served as professor in the university of Prague, since 1861 up to the end of his life — in the university of Vienna. His role in the expansion of Bolzano's doctrines among Austrian philosophers and in particular, in the Brentano school, is a disputable matter. At least personally Brentano explicitly placed responsibility for rebirth of some Bolzanian platonistic ideas among a number of his pupils on Zimmerman. I'd want to pay some attention to the role of Zimmerman in the rise of the object theory.

In course of Austrian education reform (one of renovations induced by the revolution of 1848) philosophy of Herbart was prescribed as obligatory doctrine for teaching in the universities of Habsburg monarchy. A new educational subject was incorporated into the curriculum of ober-gymnasiums, *philosophische propädeutik*, consisting of two parts: *empirical psychology* and *formal logic*. Young Zimmerman was charged to work out the textbook for the new discipline, and it was published in 1852-53, in two parts. The second part, *Formale Logik*, reproduced carefully a number of principal ideas of Bolzano's *Wissenschaftslehre*, sometimes word for word. But in one point author declines from the teaching of master, and the change became the birth of the object theory. The point is that Bolzano propounds rather theory of representations in itself which were characterized by objectness (*Gegenstandlichkeit*), than the proper object theory; i.e. his representations were divided into objectual and objectless. The latter, in turn, might be accidentally objectless (e.g. golden mountain) and in general objectless, or imaginary (e.g. round square) [5, Bd. 1, S. 297, 304–306]. But he didn't try to classify or in any way to discuss the objects of representations. It seems that just Zimmerman was first who addresses himself to tackle the matter, in the text of the second volume of the first edition of *Propädeutik*. The object of representation, he says, could be actual or non-actual; non-actual objects are of two kinds: possible and impossible. [2, S. 9]. Thus, as we can judge, it was R. Zimmerman who first tried to correlate an object to the representation of any kind (including objectless!).

The second edition of *Propädeutik* (1860) was reworked significantly. A number of Bolzano's theories and definitions were superseded by those adopted from Herbart, in particular, it does not contain discussions of representation in itself and of its object. But some another novelty was introduced there, which appeared to be very impactful in subsequence. I mean the principal characteristics, by which Zimmerman describes the concept. Those are, first, the content (what is thought in the concept), and second, the object (what concept refers to) [3, S. 19]. Content and object have nothing in common except the very fact that an object is thought by means of content. Another one remarkable point which worth to be mentioned — insistence that neither content of the concept, nor its object, are interchangeable with the word, which denotes that concept. [3, S. 24].

In the second edition the problems related to the object theory are considered in the second volume of *Propädeutik*, i.e. in *Psychology*, not in *Logic*. Howbeit, Zimmerman didn't recall here of possible and impossible objects. Nevertheless we may fix that for seven years most schoolboys over the all Austrian empire learned the logic after the first edition of his textbook and absorbed the idea of 'impossible object'. It is even more important for us here that the distinction of content and object, exercised in the second edition, appeared to be survivable, and it became afterwards one of the principal break-points in the school of Brentano.

The next crucial stage of the development of Austrian tradition in logic and philosophy was the advent of Franz Brentano, who lectured in the University of Vienna since 1874. Sometimes the very formation of the Austrian philosophy is connecting primarily with his activity. Meanwhile, as we can conclude from the above stated, the school of Brentano has not started its way from the blank space. Brentano felled into the community, members of which studied logic in the gymnasiums with the textbooks of Zimmerman, and philosophy at the universities — in the framework of Herbartian doctrines. We can cite, as an exemplary philosopher of that generation, Alois Riehl (1844–1924), who graduated from the Graz University in 1865. In full agreement with the tenets of Bolzano and Zimmerman, he believes that 'The form of science is a subject of special science, and that science is logic' [1, p. 88], and that logic is 'a theory of universal incontradictory relationships between objects in general', while the laws of thought in logical sense are 'the laws of thinkable, objectual in general' [1, p. 89].

5 F. Brentano

Franz Brentano (1838–1917), who suggested in his *Psychologie vom empirischen Standpunkt* (1874) the project of descriptive psychology, considered intentionality as an immanent property of psychical phenomena. Every mental act ought to correlate with its intentional object. While in the simplest and basic act of representation some object is thought only, than in act of judgment the object represented is affirmed as existing or is rejected as non-existing. Consequently, all judgment should be considered as existential. Brentano has demonstrated, in what way basic forms of judgments of traditional logic can be reduced to the existential form. In the result of his reduction particular judgments are being transformed into affirmative existential, and universal — into negative existential. The syllogistics constructed on the ground of this theory of judgment, consists of two rules, or forms of inference; it does not require the traditional division on figures and does not admit exactly those modes, which free logics of our days use to discard. Besides, it does not postulate that premises are to contain exactly two terms, and in general, it looks more flexible than traditional theories of syllogism. Thus, just as in the case of Bolzano, introducing of object into logic had led to the radical reforming of the latter.

Alas, just as in the case of Bolzano, the reforming did not have direct influence on the process of logic development. According to witty remark of Peter Simons, '…Brentano played Kerensky to Frege's Lenin, because when the revolution came in 1879 in the shape of Frege's *Begriffsschrift*, it involved a complete break with tradition and put Brentano's modest advance in the shade' [8, p. 42]. Yet one radical difference in views of Bolzano and Brentano formed the core for one of crucial collisions inside the school of Brentano. While distinguishing act of representation and object of representation, Brentano rejects decisively any kind of 'third entity', which could remind Bolzanian objective representation or Zimmermanian content of concept. Considerable group of his students, including A. Höfler, K. Twardowski and E. Husserl, did not take the side of master on this point, causing his great and explicit disappointment.

6 G. Frege and the school of Brentano

It is often pointed out, in the works on Frege, the striking similarities of his views on logic and mathematics with those of Bolzano; the pointing is usually followed by the ascertaining that there are no any evidences that former had ever studied the works of latter. Hereafter I'd want to specify both of the claims. First, there is a good reason to believe not only that similarity between the two authors do exist, but even more: a number of Frege's ideas look as if he was very close to the Austrian tradition in logic and philosophy. Second, there is good reason to conclude, that Frege get learned the ideas of Bolzano at least through third parties, not later than in the end of 1880s.

In his first revolutionary work of 1879, *Begriffsscrift*, Frege suggested the theory of judgment, which is strikingly relative to Brentanian one. Recall that Frege distinguished there thinkable content, which may be constituted by any combination of representations on the one hand, and the act of proper judgment, which consists in asserting of this content being thought. Surely, it is quite reasonable to assume here the influence of Herbart, whose ideas were doubtless known to Frege. The hypothesis is amplified by the fact that some other claims of Herbart are reproduced in this work almost word to word. But if we take into account that this Frege's theory of judgment is combined with breaking away the traditional decomposing of judgment on subject and predicate, than the kinship between mathematician from Jena and philosopher from Vienna looks far more persuasively. Of course, this affinity cannot prove the fact of Frege's acquaintance with *Psychology from Empirical Point of View*, but the circumstance that this kind of treatment the judgment was not practiced by anybody else except these two authors, looks remarkable.

In his next seminal work, *Die Grundlagen der Arithmetik* (1884) Frege draws strict distinction between concept and object. As it was stressed above, the accentuation on this distinction is a specific feature right of Austrian philosophical community. Of course, just fixation of this affinity cannot prove anything, but if we take into account that creating of the opus was preceded by the correspondence of the author with Brentano's pupil and colleague Karl Stumpf, than our suspicions would increase substantially. Moreover, it is well known that the very idea to expose his views on the nature of natural number and general strategy of deriving arithmetic from logic in 'prose', without using the technique of *Begriffsschrift*, was suggested to Frege by none other than Stumpf.

Finally, immediately before the termination of his work on final modification of his system, Frege examined the work of another one, not so famous member of the Brentano school, Benno Kerry (1858–1889). One of three epoch-making papers, published in the beginning of 1890s, *Über Begriff und Gegenstand*, appeared as a response to one of Kerry's critical remarks against Frege, made in his paper [7, Bd. XIII]. The work had a series of eight articles, published from 1885 to 1891. Frege is often mentioned and discussed in initial four articles, the fourth is completely devoted to the analysis of Frege's ideas. In fact, Kerry was the first Frege-inquirer[2], for he examines carefully all Frege's works, published before 1887. Taking into account the lack of interest to the ideas of Frege in that period and the frustration caused by it, I cannot believe that Frege didn't study writings of Kerry very attentively. But the latter, in the process of argumentation, refers regularly to the statements of Brentano and his disciples. But it is Bolzano whom he sites especially often and extensive. More than once he refers to Bolzano and Frege in one footnote. In a word, all this may not us assume that Frege did not have knowledge of the ideas of Bolzano, at least in the exposition of Kerry.

Moreover, the question arises, if some conclusions of another Frege's paper of that period, *Über Sinn und Bedeutung*, were inspired by analogous considerations, which he might face over and over again in the text of Kerry. I mean his splitting of *beurteilbare Inhalt* into sense and denotation. The point is that Kerry was one of those who adopted the triple of Bolzano and Zimmerman, which included not only object, but also (ideal) content of concept. Besides, as far as I know, Kerry was the first who extend the distinction up to mathematical concepts. In particular, the second article of his opus is completely devoted to considerations of the relationships between content and object of concept. For instance, he pays attention, that when some concept contains mutually exclusive features, the concept is objectless. [7, Bd. X, S. 444]. In the first article we meet the following noteworthy reasoning: "The remarkable advantage of conceptual representations against intuitive one consists in the fact

[2]It is curiously, that the second one was a student of Brentano as well, E. Husserl.

that several completely different may refer to one and the same object. Completely different concepts: ‘the chancellor of the German Reich in 1884’ and ‘the owner of Warzin in 1884’ refer to one and the same person” [7, Bd. IX, S. 460]. Really, the idea that different (contents of) concepts might correspond to the same object was rather habitual for the school of Brentano and served as a subject of a discussion in the period which immediately preceded to the appearance of Fregean theory of *Sinn* and *Bedeutung*. For example, Oskar Schmitz-Dumont in his article [9] published in the same volume of *Vierteljahrschrift* with the second article of Kerry explains that equality sign in $A = \phi(a,b)$ is justified by the fact that ‘the symbols have the same content, but the forms in which the content is expressed, are different’ [9, S. 199–200].

Another student of Brentano, Anton Marty, deserves our attention. He is identified as most probable addressee of Frege’s letter dated 29.08.1882. In the second article of his *Über subjectlosse Sätze un das Verhältniss der Grammatik zu Logik und Psychologie* (published in the *Vierteljahrschrift* again) he pays three pages of attention to discuss Fregean theory of judgment, exposed in the *Begriffsschrift* [10, S. 185–188]; in the third article of the same work he discusses the theory of denotations in terms very close to those of later Fregean. He states that there necessarily must be given some mediating link between language expression and its denotation (*Bedeutung*), which he calls an *Etymon*. He differs two functions of the sign: manifestating (*kundgebung*) and denoting, the former being primary function, while the latter secondary one [10, S. 299]. Moreover, he remarks that this mediating *Etymon* serves as ‘the way by which signs are denoting’ [10, S. 301], and expands his considerations of denoting from names to sentences (Aussagen). Marty was a true follower of Brentano and did not purport any Zimmerman-like kind of the object theory, for him the denotation of name consists in representation; his theory in general was rather psychologistic one.

7 Conclusion

We don’t have direct evidences that Frege has adopted some of his ideas from anybody from school of Brentano. But I strongly suppose that the fact that three-partial semantical structures and elements of object theory appeared at Frege's works in the beginning of 1890s, *after* his getting knowledge of the ideas of Austrian colleagues, is not contingent. Except this, it might be supposed that Frege's idea that the extension of concept is an object with equal rights as proper object, was a result of careless use of some ideas of Brentanian school.

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The logic of colors in model-theoretical and game-theoretical perspectives[1]

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abstract. This paper sketches two approaches to the color exclusion problem provided by model-theoretical and game-theoretical semantics. The case study, modeling the experimentally confirmed perception of 'forbidden' (e.g., reddish green and bluish yellow) colors, is presented as neuropsychological evidence for game-theoretical semantics.

Keywords: invariance criterion, permutation invariance, color exclusion problem, binary colors, opponent-processing model, overdefined games, non-strictly competitive games, payoff independence

1 Invariance Criterion Revisited

Logical knowledge of reality is possible since logic deals with formal, metaphysically unchanging features of reality. But what does it mean exactly? How does our formal model of reality depend on more or less sophisticated understanding of logicality?

According to Tarski's model-theoretical approach, a concept is logical if and only if 'it is invariant under all possible one-one transformations of the world onto itself' [16, p. 149][2]. Felix Klein's famous *Erlangen Program* (1872) proposed the classification of various

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[2]According to Tarski-Sher's criterion, it is better to discuss 'isomorphisms' (or 'bijections') and 'structures' instead of 'permutations' (or 'transformations') and the 'world'. This criterion is historically traced to Lindström's (1966) generalization of Mostowski's approach (1957) (see [15]).

geometries according to invariants under suitable groups of transformations. Klein pointed out that each geometric field can be characterized by the invariance condition satisfied by its notions. Tarski's criterion of logicality extended this idea to the domain of logic. Permutation invariance takes all one-one transformations into account and as a result, characterizes, according to Tarski, the most general notions. For Tarski, the science which studies these notions is logic. If we interpret the formality of a theory as its invariance under permutations of the universe it means that the theory does not distinguish between individual objects and characterizes only those properties of a model which do not depend on its nonstructural transformations. Formal property should be preserved under the arbitrary switching of individual objects. For example, 'red' and 'green' are non-formal properties, since they distinguish between things which are red and green.

However the standard argument in favor of invariance under permutation, which relies on the generality of logic, may be challenged. Tarski considered the class of permutations as the most general class of nonstructural transformations, since permutations do not respect any extra-structure. On the contrary, as Denis Bonnay pointed out, there are a lot of other concepts of similarity (i.e. approximate preservation) between structures which are far less demanding then Tarski's criterion. Thus, 'even if one grants that generality is a good way to approach logicality, there is no evidence that the class of all permutations is the best applicant for the job' [2, p. 38]. On the other hand, Ludwig Wittgenstein, for example, does not consider generality as a defining attribute of logicality; 'The mark of a logical proposition is not general validity… [18, 6.1231]. The general validity of logic might be called essential, in contrast with the accidental general validity of such propositions as 'All men are mortal' [18, 6.1232]'. Yet, what kind of general validity is essential and, as a result, logical for Wittgenstein?

2 Invariance Criterion Generalized

According to *Tractatus*, it is logically impossible for two colors to be at one place at the same time. This is because of the 'logical structure of color'. As Wittgenstein pointed out,

> 'Just as the only necessity that exists is logical necessity, so too the only impossibility that exists is logical impossibility... [18, 6.375]. For example, the simultaneous presence of two colours at the same place in the visual field is impossible, in fact logically impossible, since it is ruled out by the logical structure of colour (It is clear that the logical product of two elementary propositions can neither be a tautology nor a contradiction. The statement that a point in the visual field has two different colours at the same time is a contradiction.)' [18, 6.3751].

According to Wittgenstein, color ascriptions should be elementary. But, as the concluding remark implies, they cannot be elementary; the color ascriptions are logically interdependent, and Wittgenstein said that elementary propositions are independent. This is a well-known problem of color exclusion.

In *Some Remarks on Logical Form* Wittgenstein offered a solution to this problem. Here he is interested in examining what he calls the 'logical structure' or the 'logical form' of the 'phenomena'. As he says, 'we can only arrive at a correct analysis by, what might be called, the logical investigation of the phenomena themselves, i.e., in a certain sense a posteriori, and not by conjecturing about a priori possibilities' [19, p. 163].

A posteriori color-incompatibility claims don't express experience in its usual sense. These tautologies are logically valid due to the geometrical organization of color space. However, unlike Kant, this appeal to geometry does not entail the synthetical character of the corresponding statements. The point is that color space is a 'space of possibilities' which is for Wittgenstein a logical space.

If our logic takes into account a spectrum of invariance which preserves several additional structures, for example, a structure of color space, we may get various types of logical invariance. Therefore, following Wittgenstein we turn back from Tarski's permutation invariance criterion to Klein's original program. From the point of view of Klein's ideology, the logic of colors may be considered as a member of a family of various logics whose notions are invariant for one-one transformations which respect additional formal structures, in particular, the formal relations of colors. The invariance criterion which is generalized in this way is wide enough to include not only one extreme type of invariance (i.e. permutation invariance), but a variety of invariances which respect different types of ordering of the universe (see also [17, p. 320]).

3 Wittgenstein's 'puzzle proposition'; meaning postulates or mapping functions?

Now the key question is the following: Why did Wittgenstein consider relations between colors as formal, logical ones? My main concern is to clarify Wittgenstein's 'puzzle proposition' from *Remarks on Colour* that 'there can be a bluish green but not a reddish green'.

In his famous paper *Reds, Greens, and Logical Analysis* Hilary Putnam pointed out, that Wittgenstein's 'puzzle proposition' is analytic, in the sense in which 'analytic' means 'true on the basis of definitions plus logic'. He proposed to define the second-level predicates '$Red(F)$' (for 'F is a shade of red') and '$Grn(F)$' (for 'F is a shade of green'). In defining these predicates we must be restricted, in particular, by the postulate: 'Nothing can be classified as both a shade of red and a shade of green (i.e., 'that shade of red' and 'that shade of green' must never be used as synonyms)' [12, p. 216]. Putnam's approach to color-incompatibility has gained widespread acceptance among recent writers on perception. As Larry Hardin says in *Color for Philosophers*, 'Perhaps not being red is part of the concept of being green. Yet it seems that all a normal human being has to do to have the concept of green is to experience green in an appropriately reflective manner' [5, p. 122] (see also [22]).

Nevertheless, the introduction of certain meaning postulates seems to be irrelevant to the exegesis of Wittgenstein's ideas. The meaning postulates expand a family of analytic truths by means of dictionary conventions. On the contrary, for Wittgenstein, internal relations of colors are elementary (see, e.g. [20, § 80]). His 'puzzle proposition' is 'in a certain sense a posteriori' and its necessity does not rely on the nature of colors or 'normal human beings', but on the structural relations within the system of colors, i.e. on the geometry of colors. The objective basis for the necessity of the color-incompatibility claims is the geometry of color space as 'part of the method of projection by which the reality is projected into our symbolism' [19, p. 166].

Contrary to the meaning postulates approach, Jaakko Hintikka and Merrill Hintikka proposed to represent the concept of color 'by a function c which maps points in visual space into a color space. Then the respective logical forms of 'this patch is red' and 'this patch is green' would be $c(a) = r$ and $c(a) = g$, where r and g are the two separate objects red and green, respectively. The logical incompatibility of the two color ascriptions is then reflected according to Wittgensteinian principles by the fact that the colors red and green are represented by different names. And if so, the two propositions are logically incompatible in the usual logical notation. Their incompatibility is shown by their logical representation: a function cannot have two different values for the same argument because of its 'logical form', i.e., because of its logical type' [6, p. 161]. As Jaakko Hintikka pointed out, 'nonlogical analytical truths sometimes turn out to be logical ones when their structure is analyzed properly' [8, p. 52].

Now here is a new puzzling question; is it possible to generalize Hintikka's approach on binary colors, e.g., on reddish green or bluish yellow?

4 The opponent-processing model of binary colors vision

We perceive many colors to be binary — purple, for example, as a mixture of blue and red. We may see bluish red, but it seems impossible to experience a color that would be described as a 'reddish green' or a 'bluish yellow' . Thus, certain antagonistic pairs of colors seem not to be combined to form a binary color.

According to the opponent-processing model of colors which goes back to Ewald Hering's opponent process theory (1878), there are different types of retinal photoreceptors with optimal spectral sensitivity to specific wavelengths. Activity in any one type laterally inhibits the activity of neighboring receptors of the same type (e.g., short, middle or long wavelength receptors). Signals from the cones are assumed to be combined in an opposing fashion to produce opposing signals in retinal ganglion cells. This means that the cells are excited by the presentation of a given color and inhibited by presence of its antagonist. Red-green and blue-yellow are supposed to be spectrally opposing channels. Thus, it would be impossible for a human observer to perceive both red and green (blue and yellow) simultaneously, as that percept would require the simultaneous transmission of positive and negative signals in the same channel. As red cancels green and blue cancels yellow, reddish green and bluish yellow are considered to be ‘ forbidden’ binary colors by the opponent-processing model.

The most surprising results in modern neuropsychological literature on color vision are reports that reddish green and yellowish blue colors can be perceived (see, for example, [1] and [3]). In violation of the classical opponent-processing model, ‘stabilized-image’ experiments have shown that by stabilizing the retinal image between an antagonistic pair of red/green or blue/yellow bipartite equiluminant fields the entire region can be perceived simultaneously as both red and green (blue and yellow) or, to be more precise, as a ‘forbidden’ homogeneous mixture color whose red and green (blue and yellow) components were as clear as, for example, the green and blue components of aqua.

The first attempt at modeling these opponency violations by Hewitt Crane and Thomas Piantanida was based on the hypothesis that there is an extra stage of cortico-cortical rather then retino-cortical visual processing, i.e. a non-opponent filling-in mechanism [3, p. 1079]. The game-theoretical approach allows us to offer a uniform explanation both to standard opponent perception and to its violations in ‘stabilized-image’ experiments.

5 ‘Forbidden’ binary colors as evidence for game-theoretical semantics

From the very beginning, the opponent-processing model of colors developed in the game-theoretical framework. It suggested that the basis for color sensations lies in a process of winner-take-all competition between red and green (blue and yellow). Now it is clear that this model must take into account the competitive interactions between teams of color-labeled wavelength-selective cells. As Vincent

Billock, Gerald Gleason and Brian Tsou pointed out, 'Recent models of cortical color processing suggest that cortical color opponency may not be based on hard-wired wavelength opponency within a single cell but rather on (potentially fragile) interactions between cortical color-sensitive cells' [1, p. 2399]. They assumed that the struggle between red- and green- (blue- and yellow-) labeled units is simply blocked by the border synergy of equilumininance and stabilization [1, p. 2401].

I suppose that there is no need to block the game processing as a whole, as this synergistic effect may be captured by the game-theoretical notion of *payoff independence* introduced by Ahti-Veikko Pietarinen (see [13]). The main idea of my proposal is the interpretation of opponency violations as payoff independence in 'stabilized-image' games between red/green or blue/yellow teams of cortical color-sensitive cells. In winner-take-all games, the following holds; if there is a winning strategy of the red team then there does not exist a winning strategy of the green team, and vice versa. In 'stabilized-image' games the information exchange between the opponent teams is blocked by the synergy of equilumininance and stabilization on the cortical strategic meta-level. Consequently, both red and green (blue and yellow) teams have winning strategies in these games. In other words, 'stabilized-image' games are *over-defined*. Thus, the law of non-contradiction fails in the generalized logic of colors allowing the simultaneous perception of antagonistic pairs of colors. In contrast to winner-take-all games, 'stabilized-image' games are *non-strictly competitive* (on over-defined and non-strictly competitive games see papers by Ahti-Veikko Pietarinen and Gabriel Sandu, e.g. [14]).

Evidently, the process of 'negotiations' between teams of opponent colors is nonlinear and gradual. As shown by Billock, Gleason and Tsou, transparency and gradient effects preceded perception of homogeneous 'forbidden' colors. Their experiments also illustrated an entirely novel percept (4 out of 7 subjects) in which the red and green (or blue and yellow) bipartite fields abruptly exchange sides (one subject saw a 90° reorganization of the bipartite fields) [1, pp. 2398–2399]. These experimental data indirectly confirm Wittgenstein's statement about different types of space[3]. Switching effects in 'stabilized-image' experiments lead to simultaneous or serial reorganizations of both visual and color spaces. Whereas Wittgenstein clearly does not think that the science, and particularly neuroscience, is relevant to the resolution of philosophical problems, sometimes neuropsychological experiments influences our colors geometry, which, in turn, constitutes what the colors are. Perhaps, tomorrow the invention of special glasses with a built-in eye tracker will make reddish green and bluish yellow common colors of our 'form of life'.

In conclusion, the basic advantage of the game-theoretical approach to the logic of colors is its procedural character. Concerning the logic of binary colors, game-theoretical models seem to be the best, since a variety of game-theoretical independences provides important insights into the theory of opponent-processing. Game-theoretical notion of strategy allows us to generalize Hintikka's approach to colors as mapping functions on binary colors, in particular, on 'forbidden' binary colors.

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The modified Ramsey theorem is not a Gödel sentence

Jaakko Hintikka

abstract. Gödelian sentences are self-referential first-order sentences in the language of arithmetics. Perhaps the most celebrated one is the sentence which asserts its own unprovability. It is well known that this sentence is neither provable nor refutable in PA (Peano Arithmetics). Some logicians and philosophers have complained that such a sentence is difficult to grasp given its 'meta-theoretical' content and they started to look for undecidable arithmetical statements which have a combinatorial content. One such sentence is a variant of Ramsey's sentence: the Paris-Harrington theorem asserts its undecidability. In the present paper I shall argue that such a sentence is not first-order expressible and thereby it does not provide the desired example of a combinatorial, undecidable arithmetical sentence. Instead I shall argue that it is expressible in Independence-friendly (IF) logic.

Keywords: Peano Arithmetics, Gödel's Incompleteness theorem, undecidability, Ramsey theorem, IF logic

Mathematicians, logicians and philosophers have been puzzled by Gödel's first incompleteness theorem ever since it was published. What kinds of limitations on our logic does it reveal? If not all arithmetical truths are provable from Peano axioms by means of first-order logic, what additional resources should we resort to in arithmetic and in mathematical reasoning in general?

One way of trying to answer such questions seems to be to see what the Gödelian true but unprovable sentences are like and how their truth can be established. Gödel's own proof is constructive, but the resulting true but unprovable sentences did not turn out to be interesting mathematically and did not suggest any systematic ways of proving stronger results. It seemed therefore highly interesting when Paris and Harrington (1977) [6] discovered a simple modification of the Finite Ramsey Theorem in combinatorial mathematics that could be proved by a straightforward non-finitary combinatorial argument but was not possible to prove by means of finite combinatorial methods. This result (it will be called here modified finite Ramsey theorem or MFRT) has been taken to suggest that logical arguments should be supplemented, if not replaced, in the foundations of mathematics by combinatorial reasoning.

An emphasis on combinatorial reasoning may very well be well-placed. But if so, this recommendation is not by itself a way out of the Gödelian conundrum. For one thing, the Paris-Harrington modification of FRT is not itself the kind of sentence whose existence Gödel proved. The reason is that it is not a sentence that can be formulated in the kind of language that is used in Peano arithmetic and presupposed in Gödel's theorem, that is, in the language of first-order arithmetic. The first purpose of this note is to show its logical status. Once the logic of the modified finite Ramsey theorem is cleared up, it can be seen that it illustrates certain remarkable facts about computability.

The unmodified FRT can be formulated as follows (cf. [8, pp. 363–364]):

(1) For all k, l, m, there exists n so large that: If $X = \{1, 2, \dots, n\}$ and if $[X]^k = C_1 \cup C_2 \cup \dots \cup C_l$, then there exists $Y \subseteq X$ such that $[Y] \geq m$ and $[Y]^k \subseteq C_i$ for some $i \leq l$.

Here X and Y are sets of natural numbers and i, j, k, l, m, n are natural numbers. The cardinality of any X is $[X]$. Also let N be the set of all natural numbers. For any subset Z of N, $[Z]^k$ is the set of all (unordered) subsets of Z with k members.

For the purpose of this paper, it suffices to consider the special case known as the party problem. In it a symmetric relation R — any given symmetric relation — is assumed to be defined on N. Also, $l = 2$, $k = 2$. C_1 is the set of all the pairs $\langle x, y \rangle$ of numbers such that $R(x, y)$ and C_2 of the pairs such that $\sim R(x, y)$. If we think of $R(x, y)$ as the relation of knowing each other, then the resulting 'party problem' illustration asks whether you can assume that there is a uniform set Y of m guests simply by inviting large enough a number n of guests. The unmodified FRT answers this question affirmatively.

In the MFRT, an extra requirement is imposed on Y, viz. that Y 'large' in the sense that

(2) $[Y] > min_x(x \in Y)$.

What is the logical form of this MFRT? The obvious *prima facie* answer is, assuming a fixed R,

(3) $(\forall m)(\exists n)(\forall X)((X = \{1, 2, \ldots, n\}) \supset (\exists Y)(([Y] \geq m) \wedge (Y \subseteq X) \wedge (([Y]^2 \subseteq C_1) \vee ([Y]^2 \subseteq C_2)) \wedge ([Y] > min_x(X \in Y)))))$.

On the face of things, this is not a Gödel sentence. Gödel's incompleteness theorem deals with first-order arithmetic, whereas (3) contains two second-order quantifiers. However, they range over finite subsets of N. By means of the technique Gödel used, we can express such existential quantifiers in terms of first-order quantifiers. This takes care of $(\exists Y)$, and it is easily seen that we can similarly deal with $(\forall X)$. This is undoubtedly the basis for thinking that the MFRT is a Gödel sentence.

But this is not the full story. For the MFRT is supposed to hold for any partition $C_1, C_2, \ldots$ ergo in the case of the party problem for any symmetric relation R. Hence there is in effect an additional quantifier $(\forall R)$ in (3). What is more, this additional quantifier matters, because of its relations of dependence and independence of other quantifiers. One important relation is that $(\exists n)$ must be independent of $(\forall R)$ in order for MFRT to be valid. This independence is the main insight of this paper. This dependence is the crucial fact here.

In order to prove the independence, assume that on the contrary R does depend on n. Then given n we can define R in such a way that there are no uniform subsets large enough in X. From the unmodified FRT and its proof we can see that it takes a set of at least superexponential function of m to express the size required, i.e. the size at which it is necessary that there it has a uniform subset of m numbers (See e.g. [8, p. 363]). Assume that such a subset X of N of superexponential size is given.

We can choose X as small as possible so that the cardinality of the only uniform subset Y is precisely m, i.e. $[Y] = m$. This only requires choosing the relation R in a suitable way. Suppose now that we rename (reorder) the set $X = \{1, 2, \ldots, [X]\}$. The result is structure of the same kind as before, but with a different definition of R on X, in other words, a new value of R. Otherwise, the numbering of the members of X does not enter into the MFRT. In particular, we can re-order the set X in such a way that the members of the uniform subset Y come last in the re-ordered X. It is important that we are dealing with a re-numbering of $n = [X]$ elements and hence presuppose intuitively speaking knowing n. Since $[X]$ is a superexponential as a function of m, we have

(4) $min_x(x \in Y) > ([X] - m) > m = [Y]$

But this violates the 'largeness' requirement (2). The counter-assumption is hence impossible, and R must not depend on n.

But where does the argument leave the MFRT? We know that it is valid. How is it to be expressed in the first place? We have in to amplify (3) by bringing in the quantifier $(\forall R)$ explicitly. The question is what its dependence and independence relations to the other quantifiers in (3) are. Options include the following:

(5) $(\forall m)(\exists n)(\forall R)(- - -)$

(6) $(\forall m)(\forall R)(\exists n/\forall R)(- - -)$

(7) $(\forall m)(\forall R)(\exists n)(- - -)$

Each of these formulas gives rise to a semantical game. The argument just given shows in effect that each strategy of the vertex in the game with (5) is defeated by a suitable strategy by the falsifier. Hence (5) cannot be true and consequently cannot express MFRT.

Also, (7) is weaker than (6) and weaker than the intended force of MFRT. Hence (6) shows the logical form of the theorem.

Accordingly, MFRT involves an irreducibly independent (IF) quantifier. Hence it cannot be a formula of a traditional first-order formula or equivalent to one. And since Gödel is using a received first-order language, MFRT is not a Gödel sentence.

The irreducibly IF character of MFRT has other interesting consequences and it is related to important theoretical questions. They are discussed in [2] and [3]. From (6) it is seen that n is a function of m only, $n = r(m)$. This function is not general recursive. If it were, it would be definable in terms of a finite set of equations. This set corresponds to a set of traditional first-order sentences. Derivations from these in turn correspond to computations (see [2]). In the case of f, the formula (6) can serve as one of these sentences. But it cannot if f is to be general recursive, for (6) involves irreducibly IF Skolem functions. Hence f cannot be general recursive.

Yet f is obviously computable by a mechanical process, for we can simply by going through for a given m all the possible relations R (different 'colorings') for $n = m$, $m + 1$, $m + 2$, ... Hence MFRT appears to be highly interesting even if it is not a Gödel sentence. It is a counterexample to Church's Thesis: in a pretheoretical sense computable, but not general recursive nor therefore a Turing machine computable function.

This line of thought clearly needs fuller argument than what can be given in general. The current research is not free of confusion, and even mistakes, and needs systematic scrutiny (cf. [5]).

These questions are not examined in any detail in this paper, however. Instead, we return briefly to the initial question raised in this paper. It is not a good strategy in trying to understand Gödel's first incompleteness theorem to examine particular instances of true but unprovable arithmetical sentences. What Gödel's theorem says is essentially that the set of true arithmetical sentences is not recursively enumerable. The different axiomatizations are but different methods of enumeration (see [4]). That some particular true arithmetical sentence is not provable in some particular axiomatization is hence informative only of why one particular attempted enumeration fails, not about why every such enumeration fails, that is, what Gödel's incompleteness really means.

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Von Wright's truth-logic and around

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Devoted to Georg Henrik von Wright

abstract. In this paper von Wright's truth-logic $\mathbf{T}''$ is considered. It seems that it is a De Morgan four-valued logic **DM4** (or Belnap's four-valued logic) with endomorphism e_2. In connection with this many other issues are discussed: *twin* truth operators, a truth-logic with endomorphism g (or logic **Tr**), the lattice of extensions of **DM4**, modal logic **V2**, Craig interpolation property, von Wright–Segerberg's tense logic **W**, and so on.

Keywords: Wright's truth-logic, De Morgan four-valued logic, twin truth operators, tetravalent modal logic TML, truth logic Tr, modal logic V2, von Wright–Segerberg's tense logic

1 Four-valued classical logic $\mathbf{C_4}$ and four-valued De Morgan logic DM4

Let $\mathfrak{M}_4^C$ be a four-valued logical matrix

$$\mathfrak{M}_4^C = < \{1, b, n, 0\}, \supset, \vee, \wedge, \neg, \{1\} >$$

which is obtained from the direct product of the matrix $\mathfrak{M}$ (for classical propositional logic $\mathbf{C}_2$) with itself, i.e. $\mathfrak{M}_4^C = \mathfrak{M}_2^C \times \mathfrak{M}_2^C$, where matrix operations $\supset, \vee, \wedge, \neg$ are the following:

x	$\neg x$
1	0
b	n
n	b
0	1

$\supset$	1	b	n	0
1	1	b	n	0
b	1	1	n	n
n	1	b	1	b
0	1	1	1	1

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

Note that the set of truth-values $\{1, b, n, 0\}$ is partially-ordered in the form $0 < n,\ b < 1$, i.e. n and b are incomparable. As usual

$$x \vee y =: \neg x \supset y,$$

$$x \wedge y =: \neg(\neg x \vee \neg y),$$

$$x \equiv y =: (x \supset y) \wedge (y \supset x).$$

It is well known that matrix $\mathfrak{M}_4^C$ is characteristic for calculus $\mathbf{C}_2$. The logic with the above operations is denoted as $\mathbf{C}_4$. As usual, we will denote connectives and the similar operations by the same symbols.

Then the logic with the operations $\vee, \wedge$ and $\sim$ is called four-valued De Morgan logic **DM4**, where $\sim$ is De Morgan negation: $\sim 1 = 0, \sim b = b, \sim n = n, \sim 0 = 1$ (see [5], [9]). In another terminology, **DM4** is *Belnap's four-valued logic* [3].

2 Endomorphismus in the distributive lattices

In [6] the authors point out the fact that the modal and tense operations in a number of modal and tense logics and in corresponding algebras are expressed in terms of endomorphismus in the distributive lattices.

Let us consider one-place operations g, e_1 and e_2

x	$g(x)$	$e_1(x)$	$e_2(x)$
1	1	1	1
b	n	0	1
n	b	1	0
0	0	0	0

which are the endomorphismus in the distributive lattices:

$$f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y),$$

$$f(\neg x) = \neg f(x), f(1) = 1, f(0) = 0, f(x^{\delta}) = (f(x))^{\delta},$$ [1]

where f can be any operations from g, e_1 and e_2.

3 Von Wright's truth-logic T$''$

Now in the new terms introduced above we can define Wright's truth-logic. The expansion of **DM4** by the endomorphism e_2 leads to the logic which G.H. von Wright in 1985 denoted as **T$''$LM** and called a '*truth-logic*' (see [28]). For the sake of brevity, we will denote it as **T$''$**. Here a truth-operator T is the endomorphism e_2. Note that the following important definitions hold:

$$(*)\ e_1(x) =: \sim (e_2(\sim x)) \text{ and } e_2(x) =: \sim (e_1(\sim x)).$$ [2]

It is easy to show that all four-valued $J_i(x)$-operations are definable in **T$''$LM**, where

$$J_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{if } x \neq i \end{cases} (i = 1, n, b, 0).$$

Thus, we have:

x	$J_1(x)$	$J_b(x)$	$J_n(x)$	$J_0(x)$
1	1	0	0	0
b	0	1	0	0
n	0	0	1	0
0	0	0	0	1

[1]
$$x^{\delta} = \begin{cases} x, \text{if } \delta = 1 \\ \neg x, \text{if } \delta = 0. \end{cases}$$

[2] In [19] a four-valued 'logic of falsehood' **FL4** is formalized. In our terms it is the expansion of the language of **DM4** by the endomorphism e_1. So, in virtue of $(*)$ logics **FL4** and **T$''$** are functionally equivalent.

One may easily verify that

$$J_1 =: e_1(x) \wedge e_2(x),$$

$$J_b =: \sim e_1(x) \wedge e_2(x),$$

$$J_n =: e_1(x) \wedge \sim e_2(x),$$

$$J_0 =: \sim e_1(x) \wedge \sim e_2(x).$$

Note that $e_2(x) =: J_1 \vee J_b$. Then Wright's logic $\mathbf{T}''$ is De Morgan logic **DM4** with all $J_i(x)$-operators (but, it is important, without classical negation $\neg$). Note also that in many finite modal logics the operator J_1 is the modal operator of necessity $\Box$. Then the well-known tetravalent modal logic **TML** is **DM4** with the operator $\Box$ added to its language (see especially [9][3]). So $\mathbf{T}''$ is an extension of **TML**.

Now we need some additional definitions. A finite-valued logic $\mathbf{L}_n$ with all $J_i(x)$-operators is called *truth-complete* logic, and a logic $\mathbf{L}_n$ is said to be **C**-*extending* iff in $\mathbf{L}_n$ one can functionally express the binary operations $\supset, \vee, \wedge$, and the unary negation operation, whose restrictions to the subset $\{0, 1\}$ coincide with the classical logical operations of implication, disjunction, conjunction, and negation. In virtue of result of [2] every truth-complete and **C**-extending logic has Hilbert-style axiomatization extending the $\mathbf{C}_2$. It means that Wright's $\mathbf{T}''$ logic has such an axiomatization. Moreover, it follows from [1] that we have adequate first-order axiomatization for logic $\mathbf{T}''$ with quantifiers.

It is very interesting to generalize given four-valued von Wright's logic, i.e. to consider an arbitrary finite-valued De Morgan logic with all $J_i(x)$-operators. As a result, we obtain an entirely new class of many-valued logics which I suggest to call '*Wright's many-valued logics*' and a new class algebras which I suggest to call '*Wright's algebras*'. Then again it follows from [1] that for such logics we have adequate first-order axiomatization.

[3] However, see also [5].

4 Properties of a truth-operator T and the *twin* truth operators

The following two properties of a truth-operator T are useful:

(I) $T(\sim x) \equiv \sim T(x)$

(II) $T(x) \vee T(\sim x)$ — the law of excluded middle.

Note that these two conditions are required in the Tarski's axiomatic theory of truth with a predicate symbol *True* (see [12]).

None of these conditions is fulfilled in the logic $\mathbf{T}''$. However it is interesting to consider the operations e_1 and e_2 as the twin truth operators T_1 and T_2 bearing in mind $(*)$. Then

(I') $T_1(\sim x) \equiv \sim T_2(x)$

(II') $T_1(x) \vee T_2(\sim x)$ — the law of excluded middle.

Here we must note that the main goal pursued by von Wright has been the construction of paraconsistent logic. So the choice of the operations $\sim$ and T_2 is such that the law of contradiction

$$\sim (T_2(x) \wedge T_2(\sim x))$$

is not valid in $\mathbf{T}''$. But it is interesting that this law is valid in the form

$$\sim (T_1(x) \wedge T_2(\sim x)) \quad \text{or} \quad \sim (T_2(x) \wedge T_1(\sim x)).$$

We want to stress that *von Wright's truth logic* with the twin truth operators T_1 and T_2 seems to us very interesting.

5 Logic Tr

Let us consider the expansion of **DM4** by the endomorphism g. Now the conditions (I)–(II) are fulfilled. Note that operators $\sim$ and g commute among themselves, i.e.

$$\sim g(x) \equiv g \sim (x).$$

Moreover, this allows to define the classical negation $\neg$:

$$\neg(x) =: \sim g(x).$$

We denote a *truth* logic with the set of operations $\{\vee, \wedge, \sim, g\}$ by **Tr**.

There is a very simple and nice axiomatization of this logic (see justification below), where the operation T is g:

(A0) Axioms of classical propositional logic $\mathbf{C}_2$.

(A1) $T(A \supset B) \equiv (TA \supset TB)$.

(A2) $\neg TA \equiv T\neg A$.

(A3) $TTA \equiv A$.

The single rule of inference: *modus ponens*.[4]

It is worth to mention that there is a generalized truth-value space in kind of *bilattice* (see [11]). Indeed, smallest nontrivial bilattice is just the four-valued Belnap's logic. In [8] M. Fitting extends a first-order language by notation for elementary arithmetic, and builds the theory of truth based on bilattice. This four-valued theory of truth is an alternative to Tarsky's approach.

Also in one case, Fitting extends this language by the operation '*conflation*' (endomorphism g).

6 Interrelations between $\mathbf{T}''$ and Tr

Let P_4 be Post's four-valued functionally complete logic (see [20]). The set operation R is called functionally *precomplete* in P_4 if every enlargement $\{R, f\} = R \cup \{f\}$ of the set R by an operation f such that $f \notin R$ and $f \in P_4$ is functionally complete.

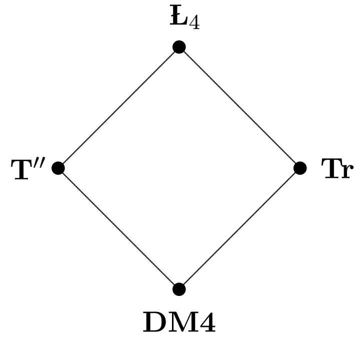
It is not difficult to prove, that the logic with the set of the operations $\{\vee, \wedge, \sim, e_2, g\}$ is four-valued Łukasiewicz logic $\mathbf{Ł}_4$ which first appeared in [15]). According to Finn's result $\mathbf{Ł}_4$ is precomplete in P_4 (see [4]). Note that in $\mathbf{Ł}_4$

[4] At the time of my report G. Sandu had asked about the logic **Tr** with the axiom $(A4)$ $TA \equiv A$. Let's denote this logic by $\mathbf{Tr^c}$. If we take the operation T as identity operation of $\mathbf{C}_2$ then the logic $\mathbf{Tr^c}$ is a conservative extension of $\mathbf{C}_2$.

$$x \vee y = max(x, y) \text{ and } x \wedge y = min(x, y),$$

i.e. the truth-values in $\mathbf{Ł}_4$ are linearly-ordered[5].

As a result, we have the following lattice of extensions of **DM4**:



7 Modal logic V2

In [25] Sobochiński presents the formula (β_2):

$$\Box p \vee \Box(p \supset q) \vee \Box(p \supset \neg q).$$

He establishes that it is not provable in **S5**, and **S5** plus (β_2) is not classical calculus $\mathbf{C}_2$. In [26] this logic is denoted by **V2**. As a consequence of Scroggs' result about *pretabularity* of **S5**[6] logic **V2** is finite-valued one. It was remarked that four-valued matrix of 'group III' from [14], i.e. matrix

$$< \{1, b, n, 0\}, \supset, \neg, \Box, \{1\} >,$$

is characteristic for **V2** (see e.g. [5, p. 190]).

In [6] it has been shown that logics **Tr** and **V2** are functionally equivalent:

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[5] In details about different finite-valued logics see in [13, ch. 5].

[6] A logic L is said to be *pretabular* if it is not finite (tabular), but its proper extension is finite. Scroggs [22] has shown that **S5** has no finite characteristic matrix but every proper normal extension does.

$\Box p =: p \wedge g(p)$,

$\Diamond p =: \neg\Box\neg p$,

$g(p) =: \Box p \vee (\neg p \wedge \Diamond p)$.[7]

Note that in [5] an algebraic semantics (named to MB-algebras) has been developed for logic **Tr (V2)**. MB-algebra is an expansion of De Morgan algebra by Boolean negation $\neg$. In this case $g(x) = \sim \neg(x) = \neg \sim (x)$. It is interesting that Pynko [21] introduces a similar algebraic structure called *De Morgan boolean algebra*. He also suggests Gentzen-style axiomatization of four-valued logic denoted by **DMB4**.

In [17] Maksimova considers all normal extensions of modal logic **S4** with the *Craig interpolation property*. From this it follows that modal logic **V2** is the *single* normal extension of modal logic **S5** with the Craig interpolation property (between **S5** and $\mathbf{C}_2$). Since the logics **Tr** and **V2** are functionally equivalent then the following theorem can be proved:

THEOREM 1. *A logic **Tr** has the Craig interpolation property.*

8 Von Wright–Segerberg's tense logic W

It is interesting that we can come to the logic **Tr** on the basis of an entirely different considerations. In [27] von Wright presents a tense logic 'And next' which deals with discrete time. In [23] Segerberg reformulates it under the name **W** and provides other proofs of completeness theorem, and decision procedure.[8]

A logic **W** is a very simple propositional logic in which a new unary operation S with the intuitive meaning of 'tomorrow' is added to the language of the classical propositional calculus. **W** is axiomatized in the following way:

(A0) Axioms of classical propositional logic $\mathbf{C}_2$.

(A1) $S(A \supset B) \equiv (SA \supset SB)$.

(A2) $\neg SA \equiv S\neg A$.

[7] However, see [23, p. 49].

[8] For detailed overview of von Wright's tense logic see Segerberg's paper [24].

The rules of inference:

R1. *Modus ponens*,

R2. *From A follows SA*.

Segerberg suggests the following Kripke-style semantics for **W** (this semantics in a simplified way is presented in [7, p. 288]). Let $N = 0, 1, 2, \ldots$ be the set of possible worlds. Valuation $v(p_i, w) = 1, 0$ ('truth', 'falsehood') for propositional variables p_i and $w \in N$. For $\supset$ and $\neg$ as usual, and for $SA : v(SA, w) = v(A, w + 1)$. Pay attention that **W** is the logic that defines the set formulas valid in N.

Concerning the logic **W** there are the following meta-logical results:

1) There is no finite axiomatization of **W** with modus ponens as sole inference rule [23].

2) Logic **W** is pretabular [7].

It is worth emphasizing that in [6] Mučnik has devised algebraic semantics for **W**, named Bg-algebras, and has proved Stone's representation theorem for them. Here it is noted that Bg-algebra with involution, where $gg(x) = x$, corresponds to the logic **V2**. Thus we again have come to the logic **Tr**.

Note than in [18] Kripke frame, consisting two possible worlds, is represented for **V2**. Here we describe Kripke frame $\imath =< T, R >$ for **W** and **Tr**, where T is the set of instants of time.

A Kripke frame $\imath =< T, R >$ is a frame for **W** if the following conditions fulfill:

1. $\forall w \in T\ \exists v \in T wRv$

   '*from every point (instant) something is accessible*'.

2. $\forall w \in T\ \forall v_1 \in\ T\ \forall v_2 \in T(wRv_1 \,\&\, wRv_2 \Rightarrow v_1 = v_2)$

   '*from every point no more than one point is accessible*'.

And for **Tr** it is necessary to add:

3. $\forall w_1 \in T\ \forall w_2 \in T\ \forall w_3 \in T(w_1 R w_2\ \&\ w_2 R w_3 \Rightarrow w_3 = w_1)$

   '*from every point in two steps we once again find ourselves in the same point*'.

THEOREM 2. *Logic* **W** + *axiom* (A3) $SSA \equiv A$ *and logic* **Tr** *are the same as the sets of derivable formulas.*

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The path of logic in Ukraine: a history of concepts

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abstract. This paper traces the development of history of logic in Ukraine in the 19^{th} century and early 20^{th} century. The author particularly discusses and compares the logical concepts of representatives of Kyiv philosophies, who made their contribution to the development of logic as a science and academic discipline. Some of them had sunk into oblivion for a long time and their names are still unknown in the logic community.

Keywords: logic, history of logic in Ukraine, Kyiv Theological Academy, St. Vladimir University of Kyiv

Nowadays one of the most important tasks for logic in Ukraine is to study the history of logical thought development. Since the research activities of logicians were always closely connected to their teaching activities it is expedient to consider history of logic as history of its teaching.

The teaching of logic in Ukraine has begun at the end of the 17^{th} century. At that time higher educational institutions founded in Ukraine were influenced by Western Europe. The Ostroh Slavic Greek Latin Academy was the first institution of such type. It was established by Prince Konstiantyn-Vasyl Ostrozhsky and Princess Halshka Ostrozhska in 1576. The Zamojska Academy was opened later in 1594.

The quality of education offered by the academies at that time was very high. The undergraduate programs were based on the European educational standards including seven courses of 'free arts' divided into trivium (grammar, rhetoric, logic) and quadrivium (arithmetic, geometry, music, astronomy). The greatest attention was paid to trivium. Logical courses (also called dialectic) were based on the European universities' programs.

However, there were practically no students from Kyiv in these academies. Many sons of wealthy Cossack families studied abroad in prestigious universities of Western Europe. Only in the 17^{th} century the first educational institution was opened in Kyiv. It was the Kyiv-Mohyla Academy, founded by Petro Mohyla, the Metropolitan of Kyiv, in 1615. The Academy has seen its golden ages from the end of the 17^{th} century and till the beginning of the 18^{th} century. With respect to teaching logic the Academy adopted methods and the curriculum of the Jesuit schools of Rzeczpospolita. The greatest philosophers of that period like Innocent Gizel (1600–1683), Stefan Iavorskyi (1658–1722) and Pheophan Prokopovich (1681–1736) gave logical lectures in the academy.

Unfortunately, after Moscow University was opened in 1755 the Kyiv-Mohyla Academy influence started rapidly to decline and in 1817 it was closed down. At the beginning of the 19^{th} century the territory of the Left-Bank Ukraine witnessed opening of several educational institutions: in 1805 — Kharkov University, in 1817 — Richelieu Lyceum and a bit later — Novorossiysk Imperial University. One more institution — Lyceum of Higher Sciences was organized in Nizhyn.

At the same time after the Kyiv-Mohyla Academy was closed down the territory of the Right-Bank Ukraine was left with no higher educational institutions until 1819 when Kyiv Theological Academy (KTA) was founded. Educational institutions of such type were not numerous at that time. In the Russian Empire there were only four Theological Academies: in St. Petersburg (since 1809), in Moscow (since 1814), in Kyiv (since 1819) and in Kazan (since 1842).

It was planned to open the Theological Academy in Kyiv as far back as in 1816. However it wasn't opened because of lack of scholars. Therefore, instead of the KTA, the theological seminary had been functioning for two years. Its major task was to prepare students as well as teachers for the new educational institution.

Kyiv Theological Academy was ceremoniously opened on September, 28, 1819 and quickly became one of the centers of classical academic education in Ukraine. It had been functioning for about 100 years and was closed down in 1920.

At that time it was very difficult to enter the academy since the prospective students had to take special exam to be accepted and logic was one of them.

During the first few years of the Academy's functioning most students were Ukrainians but with a lapse of time the Sacred Synod suggested to start admitting seminary graduates from other ecclesiastical educational regions. As a result, Russians began to prevail among students of the Academy. The 1860s saw the increase of the number of foreign students — Serbians, Bulgarians, Greeks, Romanians and Syrians. Some of them became outstanding ecclesiastical and secular figures in their countries. Greek Catholics studied at the Academy as well.

The major task for all KTA students was to fundamentally study religion through in-depth mastering of the subjects included into the Academy's curriculum, in order to attain spiritual erudition. The full academic course lasted four years and was divided into lower and higher divisions. A weekly schedule of students' classes was the following: at the lower division — Holy Scripture (2 hours), Philosophy (10 hours), General Literature (6 hours), Civil History or Mathematics (selectively) (8 hours), Greek (4 hours), Hebrew (2 hours), one of modern languages German or French (2 hours). The higher division — Holy Scripture (2 hours), Theology (12 hours), Church Literature (6 hours), Church History (6 hours), Greek (4 hours), Hebrew (2 hours), one of modern languages (2 hours).

Logic was part of the Philosophy disciplines cycle. It was taught at the lower division during the first year of studies together with the History of Ancient Philosophy and Psychology. The second year of studies contained other subjects of the Philosophy cycle: History of Contemporary Philosophy, Metaphysics and Moral Philosophy.

During one hundred years of the KTA's functioning there were about two dozens of teachers majoring in Philosophy disciplines. Those who made their contribution to the development of logic as a science and academic discipline were also among them. For example, Ivan Skvortsov (1795–1863), Vasilii Karpov (1798–1867), Iosif Mikhnevich (1809–1885), Orest Novitskyi (1806–1884), Silvestr

Hogotskyi (1813–1889), Pamfil Yurkevich (1826–1874) and Piotr Linitskyi (1839–1906). Some of them taught logic not only in KTA but also in other high institution, particularly in St. Vladimir Imperial University of Kyiv.

Although Kyiv Theological Academy played a significant role in preparing graduates, it was still an ecclesiastical educational institution. Therefore opening of secular educational establishment in the Right-Bank Ukraine, where students could chose majors in different specialties became a significant problem. Only 15 years after KTA was opened the establishment of such type was founded in Kyiv.

On November 8, 1833, Emperor of the Russian Empire expressed his will to open a new university in Kyiv. On July 15, 1834, St. Vladimir University of Kyiv accepted its first 62 students. At that time there was only one faculty — the Faculty of Philosophy. In 1835 the Faculty of Law was opened and in 1841 Medical Faculty started to accept students. In a while the Faculty of Philosophy was divided in two separate departments and such structure of the University persisted up until 1917.

By statute of 1842 the Faculty of Philosophy comprised of the First Department (History and Philology) and the Second Department (Physics and Mathematics).

The Department of Philosophy was university-wide department. Professor Orest Novitskyi was appointed as Head of the Department on the recommendation of the KTA rector, Archbishop Innocent. He became the first professor, who gave logical lectures for students of Kyiv University.

In September of 1850 teaching of Philosophy was banned following the closing of all Departments of Philosophy in universities of Russian Empire. By the way, logic as an academic discipline had not suffered any limitations then. Teaching Philosophy in secular educational institutions was limited to Logic and Psychology till 'a special command'. However, these subjects could be taught solely by theology professors.

Moreover, the Sacred Synod instructed all theological academies to develop Logic curriculum for Russian universities, and the curriculum submitted by Moscow Theological Academy was accepted. University teachers were obliged to follow it as a model [1]. Meanwhile other theological academies, including KTA, continued teaching Philosophy, Logic and Psychology according to their own curriculums.

Starting in 1850 I. Skvortsov, who was an ordinary professor of Philosophy in KTA, Doctor of Theology, an Archpriest, started to give lectures on Logic at St. Vladimir University of Kyiv. The unique document proving this has been saved in archives — Skvortsov's written consent to be appointed for the proposition by R. Tautffetter, rector of the university, to teach Logic and Psychology courses [2].

I. Skvortsov was teaching Logic until the end of 50s of the 19^{th} century. Most probably, after him Logic course were passed to N. Favorov, an ordinary professor of Philosophy, Doctor of Theology, due to his position of professor of Theology, Logic and Psychology at St. Vladimir University of Kyiv at that time.

In 1876–1887 A. Kozlov held position of a privat-docent and later extraordinary, ordinary professor at the university. It is possible to find information in the list of philosophical subjects that he gave lectures on logic (theory of induction, theory of proof and theory of scientific system) for students of Historical-Philological Department in 1885–1886 [2] following the textbook on logic written by Ziegward.

At the end of 19^{th} — the beginning of the 20^{th} century curriculum subjects of Historical-Philological Department divided into compulsive and basic disciplines; compulsive and additional disciplines; compulsive and additional subjects, depending on the additional specialty. Logic was included into the block of compulsive and basic disciplines [3]. It was taught for three hours per week.

The archives contain preserved program that was used to teach students in 1911. To get a bigger picture on the course of Logic, which was followed at the university in the beginning of 20^{th} century, we give it in full:

'Logic. Its definition and division. Historical explanation of the term. The general essay on the development of logical doctrines.

Cognitive activity. Its elements and cognition.

Representation. Psychology formation and logical significance. Historical explanation of the term.

> Notion. Psychology and Metaphysics of notion. The content and scope of the notion, its interrelation. Division of notion by content and scope. Historical essay on categories' doctrine.The basic logical rules on relations between the notions.
>
> Definition and division. Their psychology. The basic logical rules. Fallacies in definition and division. Types of division. Historical essay on definition and division.
>
> Judgement. Its psychological, grammatical, logical essence.
>
> Logical laws and their formulas. Historical essay on these laws.
>
> Argument. Its types. Psychological essence of judgement. Direct arguments. Historical essay on direct inferences.
>
> Indirect arguments. Deduction and syllogistics. The basic axiom of syllogism. Its elements. Distinction between syllogisms by the character of premises.
>
> Syllogistic figures and modus. The basic and special rules of syllogism. Conditional and dividing syllogisms. Aristotle's and Mill's doctrines on syllogism.
>
> Induction. Methods of inductive generalization. Its basic axiom. Four inductive methods. Fallacies in induction. Analogy. Its types. Historical essay on analogy.
>
> Hypothesis and proof. Its relation to inferences. Historical essay.
>
> General information about system and methods of its construction. Analysis and synthesis. Historical essay [3]'.

In the late 19^{th} century — early 20^{th} century G. Chelpanov, P. Tihomirov and V. Zenkovskyi took over teaching of Logic.

It is acknowledged that G. Chelpanov was giving lectures on Logic at the university starting from 1892 and till 1907. It can be supposed that after his departure to Moscow logical course was passed to P. Tihomirov, who was holding position of a privat-docent of the Department of Philosophy from 1907 and till 1917. It is possible to find information about the fact that logical lectures were given by P. Tihomirov in the list of philosophical and theological subjects that were taught at St.Vladimir University of Kyiv in fall semesters of 1910 [4]. P. Tihomirov recommended students the textbook on Logic by V. Minto, Th. Lipps, Ch. Sigward, A. Vvedenskyi [5].

Together with P. Tihomirov, in early 20^{th} century V. Zenkovskii was working at the Department of Philosophy. It is known that he taught Logic after teaching course of Psychology and together with course of History of Philosophy in spring semester of first year.

The teachers of Kyiv Theological Academy and St. Vladimir University of Kyiv were the cream of Kyiv philosophical community from 19^{th} and till the beginning of 20^{th} centuries. Among them were those who made their contribution to the development of logic as a science and academic discipline. The issues discussed by representatives of Kyiv philosophy were in the mainstream of the Western European philosophy of those times: how can logic be reformed? In what way shall its subject matter be determined? Can logic be considered only as a formal discipline? What are the relations between logic and psychology? Can psychology be considered as a foundation of logic? May logic be considered as a philosophical discipline? Is it necessary to include epistemological issues to the scope of logical matters? What is the methodological significance of logic? What is the significance of applied logic?

Addressing these issues, Kyiv philosophers have created original conceptions of logic which pose interest, in our opinion, not only from the point of view of history of logic but also from the point of view of current streams of the logical knowledge development.

Unfortunately, not all of them had their works published. One can learn about their views only by studying the manuscripts that had been preserved. Herewith some of the manuscripts were not written by the teachers themselves but are saved in the form of students' notes written during the lectures.

Let us attempt to enumerate representatives of Kyiv academic and university philosophy who were dealing with matter of Logic. It poses interest at least with respect to the fact that some of them had sunk into oblivion for a long time and their names are still unknown in the logical community.

The prime specialist in the field of logic in the KTA was Ivan Skvortsov, an ordinary professor of Philosophy, Doctor of Theology, an Archpriest. For more than 30 years he was delivering logical lectures at the KTA and for almost 25 years at St. Vladimir University of Kyiv. His views of logic were fully represented in his handwritten lectures on logic which are accessible today at the Institute of Manuscripts of the National Library of Ukraine. These are notes of the lectures delivered by him in 1837 written by a KTA student Andrii Monastyriov [6].

Vasilyi Karpov is another representative of Kyiv ecclesiastical academic philosophy. He was a graduate and teacher of Philosophy disciplines (including Logic) at the KTA. Later he became an ordinary professor of the St. Petersburg Theological Academy. He was devoted to teaching for over forty years (from 1825 to 1867). His textbook 'Systematic Exposition of Logic' (1856) ranks first among his works in logic [7]. It was recognized as one of the best textbooks in logic in the Russian Empire. Unfortunately, what can be found in Kyiv archives preserved only V. Karpov's personal record for the years of 1829–1830. Neither manuscripts nor lecture notes are available in Kyiv. If they do exist at all, they should be saved in St.Petersburg.

Another representative of Kyiv philosophy is Iosif Mikhnevich, a KTA graduate and an extraordinary professor. Later he became a professor of Richelieu Lyceum (Odessa), where he was teaching all Philosophy disciplines including the course of Logic. I. Mikhnevich's views of logic are expounded in his work 'An Experience of Gradual Development of Major Thinking Activity as a Guideline for Initial Teaching of Logic'(1847) [8].

From the point of view of history of logic, textbooks of Orest Novitskyi are of great interest. Or. Novitskyi was Master of Theology and Philology, an Extraordinary Professor at the KTA, the first Ordinary Professor of Philosophy and the Dean of the first Department of the Faculty of Philosophy of St.Vladimir University of Kyiv. In 1844 his 'Compendium of Logic with Preliminary Outline of Psychology' [9] was published [6]. Besides this work, issues of logic were raised in Novitskyi's short manuscript 'Something about Logic from Novitskyi's Lectures' [10], which is accessible at the Institute of Manuscripts of the National Library of Ukraine.

Views of logic of another representative of Kyiv philosophy Silvestr Hogotskyi, a KTA graduate and later an ordinary professor of Philosophy, merited ordinary professor of St.Vladimir University of Kyiv at the department of Philosophy, Doctor of Philosophy and Ancient Philology, are expounded in his 'Philosophy Lexicon' (1857–1873). It was the first in Russia four-volume Philosophy Encyclopedia [11].

A substantial hand-written heritage in logic has been left by another KTA ordinary professor, who later became a professor of Moscow University, Pamfil Yurkevich. His manuscripts are available at the Institute of Manuscripts of the National Library of Ukraine. Among them 'Curriculum and Readings in Logic' [12], 'Readings in Logic (abridged)' [13], 'Trandelenburg's Research in Logic, Abridged' [14], 'Lectures in Logic' [15], notes with regard to H.X.W. Ziegward's 'Logic' [16], 'From Logic' (lithographic lectures) [17].

The last representative of Kyiv ecclesiastical academic philosophy, who was teaching Logic at Kyiv Theological Academy, was Piotr Linitskyi, the KTA Merited ordinary professor at the Department of Logic and Metaphysics, Doctor of Theology. He left both published works and manuscripts in logic. Among his published works one can find 'Adolf Trandelenburg's Research in Logic, translated by Korsh' (1868) [18], 'On Forms and Laws of Thinking' (1895) [19], etc. Handwritten lectures in Logic by P. Linitskyi are accessible today at the Institute of Manuscripts of the National Library of Ukraine named after V.S. Vernadskyi. These are notes of the lectures delivered by him in 1889–1890 academic years written by his student P. Kudriavtsev [20].

George Chelpanov (1862—1936) arrived at Kyiv in 1892. He held position of Head of the Department of Philosophy at St. Vladimir University of Kyiv and began to teach Logic, Psychology and Philosophy. In 1907 he returned to Moscow, where became a professor of Moscow University. His logical views were presented in a well known textbook on Logic, which had been reprinted eight times before 1917 [21].

It is interesting that each of the representatives of Kyiv academic and university philosophy has been developing different conceptions of logic. They were united only by the fact that each of them tried to solve the problem of reforming logic.This problem was becoming a topical issue. Kyiv philosophers were actively participating in the discussions on how to reform the discipline and each was willing to propose an option.

The first research program for logic reforming was proposed by I. Skvortsov. Unlike Kh. Baumeister, who focused on artificial speculative logic, boiling down practical logic to a set of trivial techniques used in everyday life, I. Skvortsov was more original. He suggested not the logic of pure thinking but the logic oriented at a human being including 'metaphysical ontology, i.e. epistemology' and methodology of scientific cognition. This approach leads to two results.

Firstly, I. Skvortsov suggests considering logic together with epistemology. As a rule, philosophers consider laws of thinking and laws of cognition separately. The first one relates to logic, whereas the second relates to metaphysical ontology, or epistemology. However, in I. Skvortsov's opinion, due to close relationship between thinking and cognition (thinking is a way of cognition whereas cognition is a goal of thinking) both of them should be assigned to the single science, which could be called logic. Precisely this science investigates the required and universal forms and laws of thinking and leads to correct and profound cognition.

Secondly, the philosopher maintains practical significance of logic as a science. In his opinion, in order to learn how to think and cognize the objective world correctly, it is necessary not only to know forms and laws of thinking but also to be able to apply one's knowledge in practice.

As a result, I. Skvortsov proposes the following division of his logic. The first part is the logic of reason or theory of thinking. Here it deals with forms of thinking: notion, judgment and argument. The second part is the logic of mind or theory of cognition, which is divided into analytics of feelings, analytics of common sense and analytics of reason. The third part deals with methodology or the doctrine of application of laws and forms of thinking to the process of cognition.

In doing so, I. Skvortsov clearly separates logic per se from psychology. According to him, psychology plays the role of propaedeutics to epistemology, i.e. to the second part of his logic.

Unlike I. Skvortsov, V. Karpov believed that psychology alone may be the basis for logic. Substantiating logic on the basis of psychology he thought that within the complete system of philosophy, logic together with all other formal philosophy sciences follows psychology, it is psychology that plays the role of philosophical propaedeutics. Herewith, however, in V. Karpov's opinion, there is a risk of identifying logic with psychology insofar as both sciences study the process of thinking. According to him, the difference between logic and psychology consists in two words: being and activities.

By the way, according to V. Karpov, real sciences are those that study actual things. In addition to psychology, these are history, jurisprudence and natural history. Formal sciences are those that study relations between things. These are mathematics, grammar and logic. Since all real sciences cannot exist without expressing their content in certain forms of thinking, it is logic that plays the role of a formal instrument of cognition, although it cannot enrich a researcher with the knowledge of real life facts.

So, V. Karpov defined logic as a science specifying what forms our thinking can take driven by the aspirations of the forces of our soul that are trying to cognize any object and reveal this cognition.

The basis for such logic is psychology which studies internal side of a spiritual characteristic of being. This spirit's activity is the subject-matter of logic.

Thus one can believe that V. Karpov adhered to the standpoint of psychologism in logic. However, his psychologism is not of empirical but of speculative or even theological nature.

Proceeding from the aforementioned facts, one can understand why V. Karpov divides his textbook on logic into three parts. The first part is psychological one. Here the author deals with the psychological basis of logic. The second part is logical proper where author depicts forms of thinking. And the third part is methodological, which is devoted to the issues of application-oriented, practical logic.

Comparing to the reformist programs of I. Skvortsov and V. Karpov, the program of I. Mikhnevich demonstrates certain new features. I. Mikhnevich published his works in logic after he had retired from professorship at the KTA and moved to Richelieu Lyceum in Odessa. His program of logic reform was influenced by the lyceum specialization, which was a secular, philologically-oriented educational institution. This circumstance is most likely to be the reason for his understanding of logic as propaedeutics to philology but not to philosophy.

Comparing grammar and logic, i.e. a form of word and form of thought, I. Mikhnevich states that grammar teaches us to write and speak correctly, whereas logic teaches us to think and reason correctly. Logic is the basis for Grammar whereas Grammar is a supplement to Logic.

In view of this, I. Mikhnevich proposes the following division of logic. Firstly, it is a study on composition and formation of notions, propositions and inferences. Secondly, it is a study on the ways of connecting thoughts.

Or. Novitskyi is another representative of Kyiv ecclesiastical academic philosophy who attempted to reform logic and creatively approached to its renovation. A KTA graduate and teacher, later he moved to St.Vladimir University of Kyiv and began to teach a course of logic there.

This is how the subject matter of logic was defined by Or. Novitskyi. Logic is a science of laws on the basis of which thinking shall process the ideas and notions accumulated in memory and apply them to objects for the purpose of perceiving their essence. As can be seen, Or. Novitskyi's definition radically differs from those considered above. While his predecessors focused on studying the process of thinking, here the process of cognition is being dealt with.

In order to study laws of thinking, one should become familiar with actions and abilities of a soul that are studied by psychology. However, Or. Novitskyi was already interested not in speculative (theoretical) psychology but in empirical (experiential) psychology, which has just appeared in West European philosophy. In his opinion, logic (together with ethics and aesthetics) is a part of experiential psychology.

Unlike his predecessors, Or. Novitskyi was consistently upholding the view of the need for integrating metaphysical and logical analysis of thinking. Logic must, was supposed, to become metaphysical whereas metaphysics must become logical.

Proceeding to the issue of logic division, the philosopher dwells on the issue of natural thinking and artificial one, i.e. scientific thinking. He treats as natural such thinking which is subordinated to laws dictated by nature. Artificial thinking is the thinking that elaborates scientific cognition subject to certain rules.

Accordingly, Or. Novitskyi divides logic into pure logic, i.e. the science of laws of natural, universal thinking and applied logic or methodology. Pure logic studies laws and forms of correct thinking, whereas applied logic deals with the fundamental methods scientific cognition is subordinated to. According to Or. Novitskyi, these are mathematical, systematic and historical methods.

One more Kyiv philosopher, S. Hohotskyi, was reforming logic under the slogan 'Back to Aristotle!' He believed that the way out of the deadlock for logic was reached and suggested returning to the meaning imparted by Aristotle. It was an appeal to start understanding logic as an instrument of thinking.

P. Yurkevych proposed an absolutely new high-principled approach to logic reforming, distinct from that of his predecessors. He directs his search at the logic that is able to assimilate experience and be not just perceived as the basis for speculative constructions. He believes that the teaching of thinking in logic is closely related to the teaching of cognition. Logic must show what objective knowledge is conditioned by; applying what forms and laws a cognizing spirit switches from a subjective perception to objective cognition and how knowledge of an object is enhanced not on a mental basis but on a subject matter one.

The curriculum of Yurkevich's course in logic was not the standard and traditional one although he could not deviate far from the Sacred Synod's generally accepted instructions pertaining to the teaching of logic. His works were distinguished by in-depth knowledge of topic and conflicting approaches to logical knowledge in the middle of the 19^{th} century. It was with an overview of those approaches that he started his course in logic.

The first thing that immediately catches one's attention when start examining P. Yurkevich's Kyiv lectures in logic is the fact that he begins his course with the differentiation of formal and dialectic logic. None of the KTA professors that taught logic ever mentioned that.

P. Yurkevich highly appreciated the methodological significance of logic as a science. He believed that logic was the basis not only for theoretical but also for experiential sciences. Firstly, forms of connecting notions are absolutely identical in all sciences and these forms do not depend on the content of scientific reasoning. Secondly, the task of logic revolves around discovering the laws and norms that reveal the idea of truth.

Truth can be formal and material. Formal reflects the conformity of an object of reality with the subject matter of thought, whereas material refers to consistency of thoughts. The subject matter of logic is primarily formal truths pertaining to the correctness of reasoning but not the content thereof.

Logic is especially important for philosophy, since within philosophy perfect thinking is indispensable which makes logic a heart of philosophy.

The Kyiv philosopher emphasized the difference between logic and psychology. Psychology should study thinking as a phenomenon with all its random characteristics, which are irrelevant for the objective reality cognition. It is the task of logic to discover fundamental laws, norms, and forms of thinking and to show thinking not as it is but as it should be.

Is not it true that most of provisions of P. Yurkevich's concept can be found in modern textbooks on logic as well? In our opinion, out of all Kyiv philosophers he came closer to modern interpretation of logic as a science comparing to the others.

In the late 19^{th} and early 20^{th} centuries logic at the KTA was taught by P. Linitskyi. He had never set the task of developing a consistent concept of logic but just attempted to discover the general foundation for this science. 'Logic is a philosophical science and in the area of philosophy one of the most important matters is the issue of general foundation and origins' [15].

Why does the philosopher consider Logic as a philosophical science? In P. Linitskyi's opinion, thinking is a common element in all kinds of cognition. However, in the purest form it is displayed in philosophy. Therefore one of the major tasks of philosophy is research into the process of thinking. Logic is neither a part of psychology nor a science of philology but merely a branch of philosophy, a major propaedeutic disciplin. Herewith, in the philosopher's opinion, logic should be not only a formal science. It is a science that deals with cognition in general as well. It is impossible to perceive the nature of thinking without taking into account its goal which consists of cognitive activity of thinking.

G. Chelpanov believed that the main task of cognition is achievement of the truth by thinking. Logic is the science which considers how thinking should occur in order for the truth to be reached. The process of thinking which allows a person to reach the truth, Chelpanov calls valid thinking. Consequently Chelpanov perceives logic as a science about laws of valid thinking or in other words a science about laws which govern valid thinking.

He draws a clear distinction between Logic and Psychology. Thinking can be viewed from two standpoints: as a process, investigated by certain laws, and as a method of reaching the truth. The first standpoint relates to Psychology, the second one — to Logic. This is what defines a difference between them. Psychology is a descriptive science and explains how thinking processes occur. Logic is a normative science and considers norms and laws, which govern valid thinking.

Therefore, it can be stated that in the 19^{th} century – early the 20^{th} century logical issues were actively developed by the representatives of Kyiv Theological Academy and St.Vladimir University of Kyiv. The professors of the Departments of Philosophy have created the innovative concepts of logic, which formed the ground for teaching courses for students of these educational institutions.

However, it should be noted that identification of any tradition or approach incorporating views of Kyiv logicians is impossible. Original author's approaches are among the main reasons for this situation with perceiving of logic.

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First-order logics of branching time: on expressive power of temporal operators[1]

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abstract. We consider the logic **QCTL**, a first-order extension of **CTL** defined as a logic of Kripke frames for **CTL**. We study the question about recursive enumerability of its fragments specified by a set of temporal modalities we use. Then we discuss some questions concerned axiomatizability and Kripke completeness.

Keywords: non-classical logic, temporal logic, branching time logic, first-order logic, recursive enumerability, Kripke completeness

1 Introduction

There are at least two reasons to study branching time logics: philosophical and originating in computer science. Such logics provides us with formalisms allowing to construct and verify sentences about indeterminate future (philosophical aspect) or about some state transition systems (in computer science).

There are a lot of *propositional* temporal logics, and they found their applications both in philosophy, and computer science, see [4]. Here we deal with *first-order* logics of branching time, more exactly, first-order extensions of the logic **CTL** introduced by A. Prior, see [10].[2] It is known that such logics are undecidable and even not

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[2]The abbreviation 'CTL' means 'computational tree logic'. In [10] A. Prior does not introduce this abbreviation but he discuss logics of branching time, in particular, Cocchiarella's tense-logic (which may have the same abbreviation).

recursively enumerable; moreover for correspondent proofs it is sufficient to use only unary predicate letters, see [5, 6, 12].

The main aim of the paper is to show how one may prove that a logic (a fragment of a logic) is not recursively enumerable.

To do this we simulate positive integers with the relation 'less than'; this is the key part of the paper. Then, we use positive integers to embed the finite model theory (which is not recursively enumerable) into first-order branching time logic.

On the one hand, as a result we obtain that many fragments of logics we consider are not finitely (and even recursively) axiomatizable. Note that we define these logics semantically by means of Kripke frames; therefore, on the other hand, it follows that many calculi are not Kripke complete.

Note that the results (as theorems) presented in this paper are quite expected; moreover, most of them are known or follow from other known facts. The feature of our proofs is that, in fact, we use only embeddings of logics and nothing more. Therefore, to understand our proofs it is sufficient to be familiar with the classical first-order logic (theory) of finite domains.

2 Definitions

To define the logic we deal with, first of all we need a language. Consider the language containing

- individual variables $x_0, x_1, x_2, \ldots$;
- predicate letters P_i^m, for every $m, i \in \mathbb{N}$;
- logical constant $\bot$;
- logical connectives $\wedge$, $\vee$, $\rightarrow$;
- quantifier symbols $\forall$, $\exists$;
- modality symbols $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, $\boldsymbol{U}$;
- symbols (,), and comma.

Now define formulas we consider here. Atomic formulas are $\bot$ and $P_i^m(x_{k_1}, \ldots, x_{k_m})$ where $m, i, k_1, \ldots, k_m$ are positive integers.

If φ and ψ are formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $\forall x_i \varphi$, $\exists x_i \varphi$, $\boldsymbol{AX}\varphi$, $\boldsymbol{EX}\varphi$, $\boldsymbol{AF}\varphi$, $\boldsymbol{EF}\varphi$, $\boldsymbol{AG}\varphi$, $\boldsymbol{EG}\varphi$, $(\varphi \boldsymbol{AU} \psi)$, and $(\varphi \boldsymbol{EU} \psi)$ are formulas, too. We call such formulas *temporal*.

Also we use $\top$, $\neg$, and $\leftrightarrow$ as usual abbreviations:

$$\begin{array}{lcl} \neg\varphi & = & (\varphi \to \bot); \\ \top & = & \neg\bot; \\ (\varphi \leftrightarrow \psi) & = & ((\varphi \to \psi) \wedge (\psi \to \varphi)). \end{array}$$

We omit parenthesis that can be recovered according to the following priority of the connectives: unary modalities, quantifiers, $\neg$, binary modalities, $\wedge$, $\vee$, $\leftrightarrow$, $\to$.

We make a remark about modalities used in formulas. Any 'atomic' modality consists of two symbols: the first symbol is $\boldsymbol{E}$ or $\boldsymbol{A}$ and the second one is $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, or $\boldsymbol{U}$. Every of these symbols corresponds to some modality in more general language, see [2]. The intending meaning of the modalities $\boldsymbol{E}$, $\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, $\boldsymbol{U}$ is as follows: let us imagine that we are in some situation (current state) and it is possible to define consequences of future states; then

$\boldsymbol{E}$	means	'there is a consequence of future states (starting in the current one) such that…';
$\boldsymbol{A}$	means	'for every consequence of future states (starting in the current one) it is true that…';
$\boldsymbol{X}\varphi$	means	'φ is true in the next state of the consequence';
$\boldsymbol{G}\varphi$	means	'φ is true in every state of the consequence';
$\boldsymbol{F}\varphi$	means	'φ is true in some state of the consequence';
$\varphi\boldsymbol{U}\psi$	means	'ψ is true in at least one state of the consequence and φ is true until ψ is true'.

Of course, since we use modalities only in pairs we have no formulas like $\boldsymbol{X}\varphi$ or $\varphi\boldsymbol{U}\psi$; but now we have some informal definition for our modalities. For example $\boldsymbol{AF}\varphi$ means that for every consequence of future states (starting in the current one), φ is true in some state of the consequence.

To make the meaning of the modalities more clear we need semantics. As semantics for this language we use Kripke frames and models.

A pair $\mathfrak{F} = \langle W, R\rangle$ is called *Kripke frame* if W is non-empty set and R is a binary relation on W. We call elements of W *worlds* or *states*; we call R *accessibility relation* on W. We write wRw' instead of $\langle w, w'\rangle \in R$; if wRw' we say that w' is *accessible* from w.

We may understand a Kripke frame as a structure of (branching) time where wRw' means that w' is a possible next future state relative to w.

Here we consider mainly serial Kripke frames; recall that a frame $\mathfrak{F} = \langle W, R\rangle$ is said to be *serial* if, for any $w \in W$, there is $w' \in W$ such that wRw'.

An infinite consequence $\pi = w_0, w_1, w_2, \dots$ is called a *path* in a frame $\mathfrak{F} = \langle W, R\rangle$ if, for any $k \in \mathbb{N}$, we have $w_k \in W$ and $w_k R w_{k+1}$. We assume that π_k denotes the k-th element of the path π. We say that a path π *starts in a world* w if $\pi_0 = w$.

Note that if w is a world of a serial Kripke frame $\mathfrak{F}$ then there is at least one path in $\mathfrak{F}$ starting in w.

A triple $\mathfrak{F}(D) = \langle W, R, D\rangle$ is called *predicate Kripke frame* if $\langle W, R\rangle$ is a Kripke frame and D is a map associating with every $w \in W$ some non-empty set D_w (i. e., $D(w) = D_w$) such that

$$wRw' \implies D(w) \subseteq D(w'),$$

for any $w, w' \in W$. Elements in D_w are called *individuals* of the world w, the set $D(w)$ is called *domain* of w.

Now we need a tool connecting predicate frames with our language. As such tool we use two notions: interpretation of predicate letters and interpretation of individual variables.

Let $\mathfrak{F}(D) = \langle W, R, D\rangle$ be a predicate Kripke frame. A function I is called *interpretation of predicate letters* in $\mathfrak{F}(D)$ if $I(w, P_i^m)$ is an m-ary relation on $D(w)$, for every $w \in W$ and every predicate letter P_i^m.

A tuple $\mathfrak{M} = \langle W, R, D, I\rangle$ is called *Kripke model* if $\langle W, R, D\rangle$ is a predicate Kripke frame and I is an interpretation of predicate letters in $\langle W, R, D\rangle$.

Let $\mathfrak{F}(D) = \langle W, R, D\rangle$ be a predicate Kripke frame and let w be a world in it. A function α is called *interpretation of individual variables* in a world $w \in W$ if $\alpha(x_i) \in D(w)$, for every individual variable x_i.

Note that if w' is accessible from w and α is an interpretation of individual variables in w then α is an interpretation of individual variables in w', too, because in this case we have $D(w) \subseteq D(w')$.

For any individual variable x_i, we define the binary relation $\stackrel{x_i}{=}$ between interpretations. For interpretations α and β we put

$$\alpha \stackrel{x_i}{=} \beta \quad \leftrightharpoons \quad \alpha(x_k) = \beta(x_k), \text{ for any } k \in \mathbb{N} \text{ such that } k \neq i.$$

Let $\mathfrak{F} = \langle W, R\rangle$ be a serial Kripke frame, $\mathfrak{M} = \langle W, R, D, I\rangle$ be a Kripke model on $\mathfrak{F}$. We define the truth relation 'a formula φ is true at a world $w \in W$ in a model $\mathfrak{M}$ under an interpretation α of individual variables in w' inductively (by constructing of φ):

$(\mathfrak{M}, w) \not\models^{\alpha} \bot$;

$(\mathfrak{M}, w) \models^{\alpha} P_i^m(\overline{x}) \quad \leftrightharpoons \quad \alpha(\overline{x}) \in I(w, P_i^m)$ where $\overline{x} = (x_{k_1}, \ldots, x_{k_m})$, $\alpha(\overline{x}) = \langle \alpha(x_{k_1}), \ldots, \alpha(x_{k_m})\rangle$;

$(\mathfrak{M}, w) \models^{\alpha} \varphi_1 \wedge \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \models^{\alpha} \varphi_1$ and $(\mathfrak{M}, w) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, w) \models^{\alpha} \varphi_1 \vee \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \models^{\alpha} \varphi_1$ or $(\mathfrak{M}, w) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, w) \models^{\alpha} \varphi_1 \to \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \not\models^{\alpha} \varphi_1$ or $(\mathfrak{M}, w) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{AX}\varphi_1 \quad \leftrightharpoons$ for any path π starting in w the relation $(\mathfrak{M}, \pi_1) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{EX}\varphi_1 \quad \leftrightharpoons$ there is a path π starting in w such that $(\mathfrak{M}, \pi_1) \models^{\alpha} \varphi_1$;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{AF}\varphi_1 \quad \leftrightharpoons$ for any path π starting in w there is some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_1$;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{EF}\varphi_1 \quad \leftrightharpoons$ there are a path π starting in w and $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_1$;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{AG}\varphi_1 \quad \leftrightharpoons$ for any path π starting in w and for any $k \in \mathbb{N}$ the relation $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{EG}\varphi_1 \quad \leftrightharpoons$ there is a path π starting in w such that for any $k \in \mathbb{N}$ the relation $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \varphi_1 \boldsymbol{AU} \varphi_2 \quad \leftrightharpoons$ for any path π starting in w there is some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_2$ and, for any $j \in \mathbb{N}$, such that $j < k$ the relation $(\mathfrak{M}, \pi_j) \models^\alpha \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \varphi_1 \boldsymbol{EU} \varphi_2 \quad \leftrightharpoons$ for some path π starting in w and some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_2$ and, for any $j \in \mathbb{N}$, such that $j < k$ the relation $(\mathfrak{M}, \pi_j) \models^\alpha \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \forall x_i \varphi_1 \quad \leftrightharpoons$ for any interpretation β such that $\beta \stackrel{x_i}{=} \alpha$ and $\beta(x_i) \in D(w)$ the relation $(\mathfrak{M}, w) \models^\beta \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \exists x_i \varphi_1 \quad \leftrightharpoons$ there is an interpretation β such that $\beta \stackrel{x_i}{=} \alpha$, $\beta(x_i) \in D(w)$, and $(\mathfrak{M}, w) \models^\beta \varphi_1$.

The relations 'φ is true at w in $\mathfrak{M}$', 'φ is true in $\mathfrak{M}$', 'φ is true in $\mathfrak{F}(D)$', and 'φ is true in $\mathfrak{F}$' are defined as follows:

$(\mathfrak{M}, w) \models \varphi \quad \leftrightharpoons$ $(\mathfrak{M}, w) \models^\alpha \varphi$, for any interpretation α of individual variables in w;

$\mathfrak{M} \models \varphi \quad \leftrightharpoons$ $(\mathfrak{M}, w) \models \varphi$, for any $w \in W$;

$$\mathfrak{F}(D) \models \varphi \quad \leftrightharpoons \quad \mathfrak{M} \models \varphi, \text{ for any model } \mathfrak{M} \text{ based on } \mathfrak{F}(D);$$

$$\mathfrak{F} \models \varphi \quad \leftrightharpoons \quad \mathfrak{M} \models \varphi, \text{ for any model } \mathfrak{M} \text{ based on } \mathfrak{F}.$$

Note that any world w in a model $\mathfrak{M} = \langle W, R, D, I\rangle$ may be understood as a usual model for the classical first-order language. Indeed, as such model one may take the model $\mathfrak{M}_w = \langle D_w, I_w\rangle$ where $D_w = D(w)$ and $I_w(P_i^m) = I(w, P_i^m)$, for every predicate letter P_i^m.

We define the logic **QCTL** as the set of all temporal formulas that are true in any serial Kripke frame.

The logic **CTL** is a propositional fragment of **QCTL**.

Let also **QCL** denote the classical first-order logic in the modal-free fragment of the language for **QCTL** and let $\mathbf{QCL}_{fin}$ denote the logic of all finite models, i. e., the set of classical first-order formulas that are true in any model with finite domain.

3 Some facts

In this section we just recall some 'algorithmic' definitions and facts; so, the reader may omit this section.

Let U be some universal set (for example, a set of all formulas in some language). A set X is called *decidable* if there is an effective procedure (algorithm) A such that, for any $x \in U$,

$$A(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X; \end{cases}$$

otherwise X is called *undecidable*.

A set X is called *recursively enumerable* if $X = \varnothing$ or there is an effective procedure (algorithm) A such that $A(n)$ is defined for every $n \in \mathbb{N}$ and $X = \{A(n) : n \in \mathbb{N}\}$, in other words, if there is an effective enumeration for elements of X.

Note that if a logic is recursively (in particular, finitely) axiomatizable then it is recursively enumerable because in this case it is possible to enumerate effectively all derivations, and hence, all derivable formulas.

For a logic L, let $\overline{L}$ denote the complement of L in the set of all formulas in the language of L.

Below we will use the following facts, see [1, 11]:

- the logic **QCL** is undecidable; **QCL** is recursively enumerable and $\overline{\mathbf{QCL}}$ is not;
- the logic $\mathbf{QCL}_{fin}$ is undecidable; $\overline{\mathbf{QCL}}_{fin}$ is recursively enumerable and $\mathbf{QCL}_{fin}$ is not.

A set X is called *recursively reducible* to a set Y if there is an effective procedure (algorithm) A such that

$$x \in X \iff A(x) \in Y,$$

for any x (in the appropriate universal set U).

Let X be recursively reducible to Y. It is not hard to see that

- if Y is decidable then X is decidable;
- if Y is recursively enumerable then X is recursively enumerable.

4 Main aim

Let $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EG}, \boldsymbol{AF}, \boldsymbol{EF}, \boldsymbol{AU}, \boldsymbol{EU}\}$ and let X be a set of formulas (in some language). We use the denotation $X \upharpoonright M$ for the set of formulas in X those modalities belong to the set M^* (i. e., constructed only from modalities of M).

Our main aim is to describe some (algorithmic, semantical, deductive) properties of the logic $\mathbf{QCTL} \upharpoonright M$.

Of course, it is expected that properties of $\mathbf{QCTL} \upharpoonright M$ depend on M. Indeed, it is not hard to see that $\mathbf{QCTL} \upharpoonright \varnothing = \mathbf{QCL}$ and hence $\mathbf{QCTL} \upharpoonright \varnothing$ is finitely axiomatizable. But it follows from [12] that $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AX}, \boldsymbol{AG}\}$ is not recursively enumerable, wherefore it is not finitely (and even recursively) axiomatizable.

Below we prove the following statement.

THEOREM 1. *Let M be a set of modalities including only some of* $\boldsymbol{AX}$, $\boldsymbol{EX}$, $\boldsymbol{AG}$, $\boldsymbol{EG}$, $\boldsymbol{AF}$, $\boldsymbol{EF}$, $\boldsymbol{AU}$, $\boldsymbol{EU}$. *Then the following equivalence holds:*

$$\begin{gathered}\mathbf{QCTL} \upharpoonright M \textit{ is recursively enumerable} \\ \Longleftrightarrow \\ M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\} \textit{ or } M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}.\end{gathered}$$

Then, using Theorem 1 (and its proof) we will be able to prove some statements about algorithmic, semantical, and deductive properties of some first-order extensions of **CTL**.

5 Recursively enumerable fragments of QCTL

In this section we prove the following part of Theorem 1: if $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ or $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then $\mathbf{QCTL} \upharpoonright M$ is recursively enumerable.

To do this let us observe that

(a) $\mathbf{QCTL} \upharpoonright \varnothing = \mathbf{QCL}$;

(b) for any formula φ,

$$\begin{array}{rcl}\boldsymbol{AX}\varphi \leftrightarrow \neg\boldsymbol{EX}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{EX}\varphi \leftrightarrow \neg\boldsymbol{AX}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{AG}\varphi \leftrightarrow \neg\boldsymbol{EF}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{EF}\varphi \leftrightarrow \neg\boldsymbol{AG}\neg\varphi & \in & \mathbf{QCTL}.\end{array}$$

Because of (a), the logic $\mathbf{QCTL} \upharpoonright \varnothing$ is recursively enumerable and even finitely axiomatizable. Then, because of (b), it is sufficient to prove that $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AX}\}$ and $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AG}\}$ are recursively enumerable.

To prove the last statement we use the first-order modal logics **QD** and **QS4**.

The language of **QD** and **QS4** contains the same symbols as the language of **QCL** and the modality symbol $\Box$. Also we have the following extra rule for formula constructing: if φ is a formula then $\Box\varphi$ is a formula. We call formulas in this language *modal formulas*. The notions of Kripke frame and Kripke model are the same as they are defined in Section 2.

Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame, $\mathfrak{M} = \langle W, R, D, I\rangle$ be a Kripke model on $\mathfrak{F}$. Now we define the relation 'a modal formula φ is true at a world $w \in W$ in a model $\mathfrak{M}$ under an interpretation α of individual variables in w'. To differ the truth relation for modal formulas and the truth relation for temporal formulas we use the sign '$|\!\models$' for the first of them. This relation is defined inductively (by constructing of φ) in the same way as for temporal formulas, see page 72. The cases for the atomic formulas, for the connectives $\wedge$, $\vee$, $\rightarrow$, and for the quantifiers $\forall x_i$ and $\exists x_i$ are the same (the reader just must replace '$\models$' with '$|\!\models$'). As for $\Box$, the definition looks as follows:

$$(\mathfrak{M}, w) \,|\!\models^{\alpha} \Box\varphi_1 \quad \leftrightharpoons \quad \begin{array}{l}\text{for any } w' \in W \text{ such that } wRw' \text{ the}\\ \text{relation } (\mathfrak{M}, w') \,|\!\models^{\alpha} \varphi_1 \text{ is true.}\end{array}$$

The relations $(\mathfrak{M}, w) \,|\!\models \varphi$, $\mathfrak{M} \,|\!\models \varphi$, $\mathfrak{F}(D) \,|\!\models \varphi$, and $\mathfrak{F} \,|\!\models \varphi$ are defined as on page 73.

Now we define the logic **QD** as the set of all modal formulas that are true in any serial frame and the logic **QS4** as the set of all modal formulas that are true in any reflexive and transitive frame.

It is known that **QD** and **QS4** are finitely axiomatizable and, in particular, recursively enumerable, see [3]. We are going to show that **QCTL** $\restriction \{\boldsymbol{AX}\}$ is recursively reducible to **QD** and **QCTL** $\restriction \{\boldsymbol{AG}\}$ is recursively reducible to **QS4**.

Let us define translations T_1 and T_2. Suppose a temporal formula φ does not contain modalities different from $\boldsymbol{AX}$ and its iterations; then define $T_1(\varphi)$ to be a modal formula obtained from φ by replacing every occurrence of $\boldsymbol{AX}$ with $\Box$. Let φ does not contain modalities different from $\boldsymbol{AG}$ and its iterations; then define $T_2(\varphi)$ to be a modal formula obtained from φ by replacing every occurrence of $\boldsymbol{AG}$ with $\Box$.

Observation 1. *For any temporal formula φ without modalities different from $\boldsymbol{AX}$ and its iterations, the following equivalence holds:*

$$\varphi \in \mathbf{QCTL} \iff T_1(\varphi) \in \mathbf{QD},$$

i. e., T_1 recursively reduces **QCTL** $\restriction \{\boldsymbol{AX}\}$ *to* **QD**.

Proof. Let φ be a formula without modalities different from $\boldsymbol{AX}$ and its iterations.

Let $\varphi \notin \mathbf{QCTL}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$.

In this case, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w, the following equivalence holds:

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}, w) \mid\models^{\alpha} T_1(\psi).$$

The proof proceeds by induction on constructing of ψ and we left the details to the reader.

As a result we obtain that $(\mathfrak{M}, w_0) \mid\not\models^{\alpha_0} T_1(\varphi)$ and hence, $T_1(\varphi) \notin \mathbf{QD}$.

Let $T_1(\varphi) \notin \mathbf{QD}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \mid\not\models^{\alpha_0} T_1(\varphi)$. With the same argumentation we obtain that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$, therefore, $\varphi \notin \mathbf{QCTL}$. □

Observation 2. *For any temporal formula φ without modalities different from $\boldsymbol{AG}$ and its iterations, the following equivalence holds:*

$$\varphi \in \mathbf{QCTL} \iff T_2(\varphi) \in \mathbf{QS4},$$

i. e., T_2 recursively reduces $\mathbf{QCTL} \restriction \{\boldsymbol{AG}\}$ to $\mathbf{QS4}$.

Proof. Let φ be a formula without modalities different from $\boldsymbol{AG}$ and its iterations.

Let $\varphi \notin \mathbf{QCTL}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$.

Let R^* be reflexive and transitive closure of R and let $\mathfrak{M}^* = \langle W, R^*, D, I\rangle$. Then, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w, the following equivalence holds:

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}^*, w) \mid\models^{\alpha} T_2(\psi).$$

The proof proceeds by induction on constructing of ψ and we again left the details to the reader.

In particular, we have $(\mathfrak{M}^*, w_0) \mid\not\models^{\alpha_0} T_2(\varphi)$. Because $\mathfrak{M}^*$ is a model for **QS4**, we obtain $T_2(\varphi) \notin$ **QS4**.

Let $T_2(\varphi) \notin$ **QS4**. Then there is a reflexive and transitive model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \mid\not\models^{\alpha_0} T_1(\varphi)$. Because $\mathfrak{M}$ is reflexive, it is serial. Then, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w,

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}, w) \mid\models^{\alpha} T_2(\psi),$$

and hence, $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$. Thus, $\varphi \notin$ **QCTL**. □

Because the logics **QD** and **QS4** are recursively enumerable, from Observations 1 and 2 it follows that **QCTL** $\restriction \{\boldsymbol{AX}\}$ and **QCTL** $\restriction \{\boldsymbol{AG}\}$ are recursively enumerable. In fact, even more strong result holds.

PROPOSITION 1. *The temporal logics* **QCTL** $\restriction \{\boldsymbol{AX}\}$ *and* **QCTL** $\restriction \{\boldsymbol{AG}\}$ *are finitely axiomatizable.*

PROOF. It is sufficient to observe that the translation T_1 is an isomorphism between **QCTL** $\restriction \{\boldsymbol{AX}\}$ and **QD**, the translation T_2 is an isomorphism between **QCTL** $\restriction \{\boldsymbol{AG}\}$ and **QS4**. □

6 Main technical construction

Now we start to prove that if $M \not\subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ and $M \not\subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then **QCTL** $\restriction M$ is not recursively enumerable.

Here we present some technical constructions and statements; then we apply them to achieve the main aim.

Let us fix three binary letters and one unary letter; to make it easier to understand (and with according of their intending meaning) we denote them $\approx$, $\prec$, $\prec_1$, and L. To explain the intending meaning of these predicate letters, let us imagine that we try to order equivalence classes of some set; then

$x \approx y$	means	‘x and y are equivalent’;
$x \prec y$	means	‘x is less than y’;
$x \prec_1 y$	means	‘y is the next element after x’;
$L(x)$	means	‘x is a label (for a current state)’.

With help of ‘labels’ and modalities we show how to describe the condition that equivalence classes are ordered as positive integers by the relation ‘less than’.

Let us define some formulas. The formula A_1 claims $\approx$ to be an equivalence relation:

$$\begin{array}{rcl} A_1 & = & \forall x(x \approx x) \wedge \forall x \forall y(x \approx y \to y \approx x) \wedge \\ & & \qquad \forall x \forall y \forall z(x \approx y \wedge y \approx z \to x \approx z). \end{array}$$

The formula A_2 claims $\approx$ to be a congruence relative to $\prec$:

$$A_2 \;=\; \forall x \forall y \forall u \forall v(x \approx u \wedge y \approx v \to (x \prec y \to u \prec v)).$$

The formula A_3 claims $\prec$ to be a strict linear order (on equivalence classes):

$$\begin{array}{rcl} A_3 & = & \forall x \,\neg(x \prec x) \wedge \\ & & \qquad \forall x \forall y \forall z(x \prec y \wedge y \prec z \to x \prec z) \wedge \\ & & \qquad\qquad \forall x \forall y(x \prec y \vee x \approx y \vee y \prec x). \end{array}$$

The formula A_4 defines $\prec_1$ as a successor relation with respect to $\prec$:

$$A_4 \;=\; \forall x \forall y((x \prec_1 y) \leftrightarrow (x \prec y \wedge \neg \exists z(x \prec z \wedge z \prec y))).$$

The formula A_5 means that any element has a successor:

$$A_5 \;=\; \forall x \exists y(x \prec_1 y).$$

Let $x \not\approx y$ be the abbreviation for $\neg(x \approx y)$. The formulas A_6 and A_7 claim heredity for $\prec_1$ and $\not\approx$, correspondingly:

$$\begin{array}{rcl} A_6 & = & \forall x \forall y(x \prec_1 y \to \boldsymbol{AG}(x \prec_1 y)); \\ A_7 & = & \forall x \forall y(x \not\approx y \to \boldsymbol{AG}(x \not\approx y)). \end{array}$$

The formula A_8 means for a world that it has a unique label (if it has a label at all):

$$A_8 \;=\; \forall x \forall y(L(x) \wedge L(y) \to x \approx y).$$

The formula A_9 means that any next world has the next label:

$$A_9 \;=\; \forall x \forall y(L(x) \wedge x \prec_1 y \to \boldsymbol{AX} L(y)).$$

Let

$$A \quad = \quad \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_9).$$

Let also

$$B \quad = \quad \exists x(L(x) \wedge \neg\exists y(y \prec x)).$$

The formula B means that there is a least element ('zero') and it is a label (in a world where B is true). Finally, let

$$C \quad = \quad \forall x \boldsymbol{EF} L(x).$$

The formula C means that any element (of some current world) labels some future world.

As we will see below, the formula $A \wedge B \wedge C$ provides us with a condition that is sufficient to prove that equivalence classes are ordered with $\prec$ as positive integers with the relation 'less than'. But before this we show that the formula $A \wedge B \wedge C$ is **QCTL**-satisfiable, i. e., $\neg(A \wedge B \wedge C) \notin$ **QCTL**.

Let $\mathfrak{M}_0 = \langle W_0, R_0, D_0, I_0 \rangle$ where

$$\begin{array}{lcl}
W_0 & = & \{w_i : i \in \mathbb{N}\}; \\
w_i R_0 w_j & \leftrightharpoons & j = i + 1; \\
D_0(w_i) & = & \mathbb{N}; \\
I_0(w_i, \approx) & = & \{\langle m, m \rangle : m \in \mathbb{N}\}; \\
I_0(w_i, \prec) & = & \{\langle m, k \rangle : m, k \in \mathbb{N} \text{ and } m < k\}; \\
I_0(w_i, \prec_1) & = & \{\langle m, m+1 \rangle : m \in \mathbb{N}\}; \\
I_0(w_i, L) & = & \{\langle i \rangle\}.
\end{array}$$

Lemma 1. *It is true that* $(\mathfrak{M}_0, w_0) \models A \wedge B \wedge C$.

Proof is straightforward and left to the reader. □

Proposition 2. *The formula* $A \wedge B \wedge C$ *is* **QCTL**-*satisfiable.*

Proof. It is sufficient to observe that the accessibility relation in the model $\mathfrak{M}_0$ is serial and then use Lemma 1. □

Now let us turn to the key technical lemmas of the article.

Let $\mathfrak{M} = \langle W, R, D, I \rangle$ be a serial model and w^* be a world in W such that $(\mathfrak{M}, w^*) \models A \wedge B \wedge C$.

For simplicity let us use the following abbreviations, for any $w \in W$:

$$\approx^w = I(w, \approx), \ \ \prec^w = I(w, \prec), \ \ \prec_1^w = I(w, \prec_1), \ \ L^w = I(w, L).$$

Let also, for any $w, w' \in W$ and any $k \in \mathbb{N}$,

$$\begin{array}{lcl} wR^0w' & \leftrightharpoons & w = w'; \\ wR^{k+1}w' & \leftrightharpoons & wR^k u \text{ and } uRw', \text{ for some } u \in W; \\ wR^*w' & \leftrightharpoons & wR^m w', \text{ for some } m \in \mathbb{N}. \end{array}$$

Observe that $(\mathfrak{M}, w) \models A_1 \wedge \ldots \wedge A_9$, for any $w \in W$ such that w^*R^*w; this is so because $(\mathfrak{M}, w^*) \models \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_9)$.

Lemma 2. *Let $w \in W$ and w^*R^*w. Then the relation $\approx^w$ is a congruence with respect to the relation $\prec^w$.*

Proof immediately follows from $(\mathfrak{M}, w) \models A_1 \wedge A_2$. □

Because of Lemma 2 we may define congruence classes: for any $w \in W$ such that w^*R^*w and any $a \in D(w)$ we put

$$[a]^w \ = \ \{b \in D(w) : b \approx^w a\}.$$

Note that the relations $\prec^w$ and $\prec_1^w$ on $D(w)$ generate the relations $\prec^w$ and $\prec_1^w$ on equivalence classes defined in the following way:

$$\begin{array}{lcl} [a]^w \prec^w [b]^w & \leftrightharpoons & a \prec^w b; \\ [a]^w \prec_1^w [b]^w & \leftrightharpoons & a \prec_1^w b, \end{array}$$

for any $w \in W$ such that w^*R^*w and for any $a, b \in D(w)$. Let also

$$[a]^w \preccurlyeq^w [b]^w \ \leftrightharpoons \ [a]^w \prec^w [b]^w \text{ or } [a]^w = [b]^w.$$

Let, for any $w \in W$ such that w^*R^*w,

$$D^w \ = \ \{[a]^w : a \in D(w)\}.$$

For the rest of this section our aim is to prove that $\langle D^{w^*}, \prec^{w^*}\rangle$ is isomorphic to $\langle \mathbb{N}, <\rangle$.

LEMMA 3. *Let $w \in W$ and $w^* R^* w$. Then the relation $\prec^w$ is a strict linear order on D^w and $\prec_1^w$ is the successor relation on D^w with respect to $\prec^w$.*

PROOF immediately follows from $(\mathfrak{M}, w) \models A_3 \wedge A_4$. □

Since $(\mathfrak{M}, w^*) \models B$, there is $a_0 \in D(w^*)$ such that $L^{w^*}(a_0)$ and $[a_0]^{w^*} \preccurlyeq^{w^*} [a]^{w^*}$, for any $a \in D(w^*)$. Then, due to $(\mathfrak{M}, w^*) \models A_5$, there are $a_1, a_2, a_3, \ldots \in D(w^*)$ such that

$$a_0 \prec_1^{w^*} a_1 \prec_1^{w^*} a_2 \prec_1^{w^*} a_3 \prec_1^{w^*} \ldots;$$

note by the way that equivalence classes generated by elements $a_0, a_1, a_2, a_3, \ldots$ in $D(w^*)$ are pairwise different.

LEMMA 4. *Let $w \in W$ and $w^* R^* w$. Then*

$$\begin{array}{lcl} a \prec_1^{w^*} b & \Longrightarrow & a \prec_1^{w} b; \\ a \not\approx^{w^*} b & \Longrightarrow & a \not\approx^{w} b, \end{array}$$

for any $a, b \in D(w^*)$.

PROOF immediately follows from $(\mathfrak{M}, w^*) \models A_6 \wedge A_7$. □

LEMMA 5. *Let $w \in W$, $k \in \mathbb{N}$, and $w^* R^k w$. Then $L^w(a_k)$ is true.*

PROOF proceeds by induction on k.

Let $k = 0$. Then we must prove that $L^{w^*}(a_0)$ is true; but we have $L^{w^*}(a_0)$ to be true because of choosing of a_0.

Let the statement be true for k; we prove it for $k+1$. Let $w^* R^{k+1} w$. Then there is $u \in W$ such that $w^* R^k u$ and uRw. By induction hypothesis, $L^u(a_k)$ holds. By Lemma 4, we have $a_k \prec_1^u a_{k+1}$. Then, from $(\mathfrak{M}, u) \models A_9$ and uRw we obtain $L^w(a_{k+1})$. □

LEMMA 6. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_k$, for some $k \in \mathbb{N}$.*

Proof. Because of $(\mathfrak{M}, w^*) \models C$, there is $w \in W$ such that $w^* R^k w$, for some $k \in \mathbb{N}$, and $L^w(b)$ is true. By Lemma 5, $L^w(a_k)$ is true. Then, from $(\mathfrak{M}, w) \models A_8$ we obtain $b \approx^w a_k$ and, by Lemma 4, we obtain $b \approx^{w^*} a_k$. □

As a corollary we obtain the next statement.

Proposition 3. *The structures $\langle D^{w^*}, \prec^{w^*} \rangle$ and $\langle \mathbb{N}, < \rangle$ are isomorphic.*

Proof. For any $k \in \mathbb{N}$, let $f(k) = [a_k]^{w^*}$. We show that f is the required isomorphism. Clearly, $k < m$ implies $[a_k]^{w^*} \prec^{w^*} [a_m]^{w^*}$. So, we must prove that f is injective and surjective.

Injectivity of f. Let $k \neq m$; without a loss of generality we may assume that $k < m$. But then $[a_k]^{w^*} \prec^{w^*} [a_m]^{w^*}$ and hence $f(k) \neq f(m)$.

Surjectivity of f. Let $b \in D(w^*)$. Then, by Lemma 6, we have $b \approx^{w^*} a_k$, for some $k \in \mathbb{N}$; this means that $[b]^{w^*} = [a_k]^{w^*} = f(k)$. □

7 Embedding of $\mathbf{QCL}_{fin}$ into QCTL

Let φ be some closed classical first-order formula and let y be some individual variable not occurring in φ. Let also $x \preccurlyeq y$ be an abbreviation for the formula $(x \prec y \vee x \approx y)$. We define the translation T_y:

$$
\begin{array}{lcl}
T_y(\bot) & = & \bot; \\
T_y(P_i^m(x_{k_1}, \ldots, x_{k_m})) & = & P_i^m(x_{k_1}, \ldots, x_{k_m}); \\
T_y(\psi' \wedge \psi'') & = & T_y(\psi') \wedge T_y(\psi''); \\
T_y(\psi' \vee \psi'') & = & T_y(\psi') \vee T_y(\psi''); \\
T_y(\psi' \to \psi'') & = & T_y(\psi') \to T_y(\psi''); \\
T_y(\forall x \psi') & = & \forall x (x \preccurlyeq y \to T_y(\psi')); \\
T_y(\exists x \psi') & = & \exists x (x \preccurlyeq y \wedge T_y(\psi'))
\end{array}
$$

where ψ' and ψ'' are subformulas of the formula φ. Then we put $T(\varphi) = \forall y \, T_y(\varphi)$.

To explain the intending meaning of $T_y(\varphi)$ and $T(\varphi)$ let us imagine that we interpret individual variables as positive integers and $\preccurlyeq$ as the relation $\leqslant$ on $\mathbb{N}$. Then $T_y(\varphi)$ means 'φ is true in any model with elements $0, \dots, y$' and $T(\varphi)$ means 'φ is true in any finite model'.

Let $P^{m_1}_{i_1}, \dots, P^{m_k}_{i_k}$ be the list of all predicate letters occurring in φ. We define the formula $Congr(\varphi)$ as a conjunction of formulas in the following form:

$$\forall x_1 \dots \forall x_{m_j} \forall y_1 \dots \forall y_{m_j} \big(\bigwedge_{i=1}^{m_j} (x_i \approx y_i) \to \\ \to (P^{m_j}_{i_j}(x_1, \dots, x_{m_j}) \to P^{m_j}_{i_j}(y_1, \dots, y_{m_j}))\big)$$

where $j \in \{1, \dots, k\}$. Let

$$Emb(\varphi) \quad = \quad A \wedge B \wedge C \wedge Congr(\varphi) \to T(\varphi).$$

Lemma 7. $\varphi \in \mathbf{QCL}_{fin} \iff Emb(\varphi) \in \mathbf{QCTL}$.

Proof. Suppose $\varphi \notin \mathbf{QCL}_{fin}$. Then there is a classical model $\mathfrak{S} = \langle S, J\rangle$ where S is a finite set, J is an interpretation of predicate letters in S, such that $\mathfrak{S} \not\models \varphi$. Without a loss of generality we may assume that $S = \{0, \dots, n\}$, for some $n \in \mathbb{N}$.

Let $\mathfrak{M}_0$ be Kripke model defined on page 81. We extend I_0 on predicate letters occurring in φ; let

$$I_0(w_0, P^{m_j}_{i_j}) \quad = \quad \{\langle k_1, \dots, k_{m_j}\rangle : \langle k_1, \dots, k_{m_j}\rangle \in J(P^{m_j}_{i_j})\},$$

for any $j \in \{1, \dots, k\}$, i.e., $I_0(w_0, P^{m_j}_{i_j}) = J(P^{m_j}_{i_j})$. Let also $I_0(w_i, P^{m_j}_{i_j}) = \varnothing$, for any $i \in \mathbb{N}^+$.

Then we claim $(\mathfrak{M}_0, w_0) \not\models Emb(\varphi)$. Since the letter $\approx$ is interpreted with the identity relation on $D(w_0)$, it is clear that $(\mathfrak{M}_0, w_0) \models Congr(\varphi)$. By Lemma 1, we have $(\mathfrak{M}_0, w_0) \models A \wedge B \wedge C$. Hence we just must prove that $(\mathfrak{M}_0, w_0) \not\models \forall y\, T_y(\varphi)$. It is sufficient to interpret y as n. More exactly, the following condition holds: for any subformula ψ of the formula φ, for any interpretation α such that $\alpha(x) \in \{0, \dots, n\}$, for any individual variable x, and $\alpha(y) = n$,

$$\mathfrak{S} \models^\alpha \psi \iff (\mathfrak{M}_0, w_0) \models^\alpha T_y(\psi).$$

We left the proof of the condition to the reader; it proceeds by induction on constructing of ψ. Thus, $(\mathfrak{M}_0, w_0) \not\models^\alpha T_y(\varphi)$, for any such interpretation α. Therefore $(\mathfrak{M}_0, w_0) \not\models \forall y\, T_y(\varphi)$ and hence $(\mathfrak{M}_0, w_0) \not\models Emb(\varphi)$.

Since the model $\mathfrak{M}_0$ is serial, we have $Emb(\varphi) \notin \mathbf{QCTL}$.

Now suppose $Emb(\varphi) \notin \mathbf{QCTL}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I\rangle$ such that $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$, for some $w^* \in W$. From $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$ we obtain $(\mathfrak{M}, w^*) \models A \wedge B \wedge C$ and hence we may use results of Section 6. Let

$$\approx^{w^*} = I(w^*, \approx), \quad \prec^{w^*} = I(w^*, \prec).$$

Let also

$$\begin{array}{rcl} [a] & = & \{b \in D(w^*) : b \approx^{w^*} a\}; \\ D^{w^*} & = & \{[a] : a \in D(w^*)\}; \\ [a] \prec^{w^*} [b] & \leftrightharpoons & a \prec^{w^*} b. \end{array}$$

Then, by Proposition 3, the structures $\langle D^{w^*}, \prec^{w^*}\rangle$ and $\langle \mathbb{N}, <\rangle$ are isomorphic. Let $g : D^{w^*} \to \mathbb{N}$ be an isomorphism between the structures. Let us define an interpretation J for predicate letters occurring in $T(\varphi)$. Let P be a predicate letter occurring in $T(\varphi)$, i. e., P is one of the letters $P_{i_1}^{m_1}, \ldots, P_{i_k}^{m_k}$, $\approx$, and $\prec$, and let m be the arity of P. We put

$$J(P) = \{\langle g([b_1]), \ldots, g([b_m])\rangle : \langle b_1, \ldots, b_m\rangle \in I(w^*, P)\}.$$

From $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$ it follows that $(\mathfrak{M}, w^*) \models Congr(\varphi)$ and hence the relation $\approx^{w^*}$ is a congruence relative to $I(w^*, P_{i_j}^{m_j})$, for any $j \in \{1, \ldots, k\}$. This means that J is well defined. Note that, in particular, $J(\approx)$ is the identity relation on $\mathbb{N}$ and $J(\prec)$ is the relation $<$ on $\mathbb{N}$.

Let $\mathfrak{S} = \langle \mathbb{N}, J\rangle$.

We claim $\mathfrak{S} \not\models T(\varphi)$. Let α be an interpretation of individual variables in $D(w^*)$ and β be an interpretation of individual variables in $\mathbb{N}$; we call α and β *agreed* interpretations if $\beta(x) = g([\alpha(x)])$, for any variable x. Then

$$(\mathfrak{M}_0, w_0) \models^{\alpha} \chi \iff \mathfrak{S} \models^{\beta} \chi,$$

for any agreed interpretations α and β and for any classical formula χ containing no predicate letters different from $P_{i_1}^{m_1}, \ldots, P_{i_k}^{m_k}$, $\approx$, and $\prec$; the details are left to the reader. Since $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$, we have $(\mathfrak{M}, w^*) \not\models T(\varphi)$, and hence, $\mathfrak{S} \not\models T(\varphi)$.

It follows from $\mathfrak{S} \not\models T(\varphi)$ that $\mathfrak{S} \not\models^{\alpha} T_y(\varphi)$, for some interpretation α. Let $n = \alpha(y)$ and $\mathfrak{S}' = \langle \{0, \dots, n\}, J' \rangle$ where J' is a restriction of J on $\{0, \dots, n\}$. Then, for any subformula ψ of the formula φ, for any interpretation β such that $\beta(x) \in \{0, \dots, n\}$, for any variable x, and $\beta(y) = n$,

$$\mathfrak{S}' \models^{\beta} \psi \iff \mathfrak{S} \models^{\beta} T_y(\psi).$$

The details of the proof are left to the reader.

Since $\mathfrak{S} \not\models^{\alpha} T_y(\varphi)$ we obtain $\mathfrak{S}' \not\models \varphi$. The model $\mathfrak{S}'$ is finite, therefore $\varphi \notin \mathbf{QCL}_{fin}$. □

Corollary 1. *The logic* **QCTL** *is not recursively enumerable.*

Corollary 2. *The logic* $\mathbf{QCTL} \restriction \{\boldsymbol{AX}, \boldsymbol{AG}\}$ *is not recursively enumerable.*

8 Modifications for other fragments

Now we have got a 'weak Theorem 1': just for the case $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EF}\}$. What about other modalities? In this section we show that the fragment of **QCTL** with any of the modalities $\boldsymbol{AF}$, $\boldsymbol{EG}$, $\boldsymbol{AU}$, and $\boldsymbol{EU}$ is not recursively enumerable. Since, for any formula φ,

$$\begin{array}{lcl} \boldsymbol{EG}\varphi \leftrightarrow \neg \boldsymbol{AF} \neg \varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{AF}\varphi \leftrightarrow \neg \boldsymbol{EG} \neg \varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{AF}\varphi \leftrightarrow \top \boldsymbol{AU} \varphi & \in & \mathbf{QCTL}, \end{array}$$

it is sufficient to prove that $\mathbf{QCTL} \restriction \{\boldsymbol{EU}\}$ is not recursively enumerable and that at least one of $\mathbf{QCTL} \restriction \{\boldsymbol{AF}\}$ and $\mathbf{QCTL} \restriction \{\boldsymbol{EG}\}$ is not recursively enumerable. We consider the first and the second fragments.

To prove that these fragments are not recursively enumerable, it is sufficient to construct some embeddings of $\mathbf{QCL}_{fin}$ into the fragments. To construct such embeddings we modify the embedding Emb defined on page 85. Our modifications do not concern 'classical' part of $Emb(\varphi)$, i. e., we modify only the formulas A, B, and C defined in Section 6. The aim is to prove a statement like Lemma 7. Observe that the formulas A, B, and C are used in the proof of Lemma 7 only once: we need just Proposition 3 to be true. Therefore for our purposes it is sufficient to modify A, B, and C so that Proposition 3 remains to be true for resulting formulas. Finally, observe that the key condition in the proof of Proposition 3 is Lemma 6.

8.1 Fragments with $\boldsymbol{EU}$

Let $\boldsymbol{AR}$ be the dual modality for $\boldsymbol{EU}$; we define it as the following abbreviation:

$$(\varphi \boldsymbol{AR} \psi) \quad = \quad \neg(\neg\varphi \boldsymbol{EU} \neg\psi),$$

for any formulas φ and ψ. Observe that, for any φ,

$$\begin{array}{lcl} \boldsymbol{AG}\varphi \leftrightarrow \bot \boldsymbol{AR} \varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{EF}\varphi \leftrightarrow \top \boldsymbol{EU} \varphi & \in & \mathbf{QCTL}, \end{array}$$

and therefore we may use $\boldsymbol{AG}$ and $\boldsymbol{EF}$ as corresponding abbreviations. Then, we may define all formulas $A_1, \ldots, A_8$, B, and C but not A_9 because A_9 contains the modality $\boldsymbol{AX}$. Instead of A_9 we take the formula A_9':

$$A_9' \quad = \quad \forall x \forall y \big(L(x) \wedge x \prec_1 y \to L(y) \boldsymbol{AR} (L(x) \vee L(y))\big).$$

Let $A' = \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_8 \wedge A_9')$.

Proposition 4. *The formula $A' \wedge B \wedge C$ is* **QCTL**-*satisfiable.*

Proof. It is sufficient to check that $(\mathfrak{M}_0, w_0) \models \boldsymbol{AG} A_9'$ and use Lemma 1. $\square$

Let us use the notations introduced in Section 6 where $\mathfrak{M}$ is a model and w^* is a world in it such that $(\mathfrak{M}, w^*) \models A' \wedge B \wedge C$. It is clear that Lemmas 2, 3, and 4 are true. Lemma 5 is not true but we will prove a similar statement.

Lemma 8. *Let $w \in W$, $k \in \mathbb{N}$, and $w^* R^k w$. Then $L^w(a_m)$ is true, for some $m \in \{0, \ldots, k\}$.*

PROOF proceeds by induction on k. If $k = 0$ then with the same argumentation as in Lemma 5 we obtain $L^{w^*}(a_0)$.

Let the statement is true for k; we prove it for $k+1$. Let $w^* R^{k+1} w$. Then there is $u \in W$ such that $w^* R^k u$ and uRw. By induction hypothesis, there is $m \in \{0, \ldots, k\}$ such that $L^u(a_m)$ holds. By Lemma 4, we have $a_m \prec_1^u a_{m+1}$. Then, from $(\mathfrak{M}, u) \models A_9'$ and uRw we obtain $L^w(a_m)$ or $L^w(a_{m+1})$. $\square$

The next lemma is like Lemma 6.

LEMMA 9. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_m$, for some $m \in \mathbb{N}$.*

PROOF. Because of $(\mathfrak{M}, w^*) \models C$, there is $w \in W$ such that $w^* R^k w$, for some $k \in \mathbb{N}$, and $L^w(b)$ is true. By Lemma 8, $L^w(a_m)$ is true, for some $m \in \{0, \ldots, k\}$. Then, from $(\mathfrak{M}, w) \models A_8$ we obtain $b \approx^w a_m$ and, by Lemma 4, we obtain $b \approx^{w^*} a_k$. $\square$

From Lemma 9 we obtain a statement like Proposition 3.

PROPOSITION 5. *The structures $\langle D^{w^*}, \prec^{w^*} \rangle$ and $\langle \mathbb{N}, < \rangle$ are isomorphic.*

PROOF. Follow to the proof of Proposition 3. $\square$

Let φ be some closed classical first-order formula. We define $Emb'(\varphi)$:

$$Emb'(\varphi) \quad = \quad A' \wedge B \wedge C \wedge Congr(\varphi) \to T(\varphi).$$

LEMMA 10. $\varphi \in \mathbf{QCL}_{fin} \iff Emb'(\varphi) \in \mathbf{QCTL}$.

PROOF proceeds with the same argumentation as the proof for Lemma 7; the difference is in use Proposition 5 instead of Proposition 3. $\square$

COROLLARY 3. *The logic $\mathbf{QCTL} \restriction \{\boldsymbol{EU}\}$ is not recursively enumerable.*

8.2 Fragments with $\boldsymbol{AF}$

Observe that we may use the modality $\boldsymbol{EG}$ as an abbreviation because $\boldsymbol{EG}\varphi \leftrightarrow \neg\boldsymbol{AF}\neg\varphi \in \mathbf{QCTL}$, for any formula φ.

Let p be a propositional variable (i. e., some 0-ary predicate letter). We need it to define 'heredity' formulas:

$$
\begin{array}{rcl}
A''_6 & = & \forall x \forall y \big((x \prec_1 y \leftrightarrow \boldsymbol{AF}(x \prec_1 y \wedge p)) \wedge \\
 & & \qquad (x \prec_1 y \leftrightarrow \boldsymbol{AF}(x \prec_1 y \wedge \neg p))\big); \\
A''_7 & = & \forall x \forall y \big((x \not\approx y \leftrightarrow \boldsymbol{AF}(x \not\approx y \wedge p)) \wedge \\
 & & \qquad (x \not\approx y \leftrightarrow \boldsymbol{AF}(x \not\approx y \wedge \neg p))\big).
\end{array}
$$

Let us also define A''_9:

$$
\begin{array}{rcl}
A''_9 & = & \forall x \forall y \big(L(x) \wedge x \prec_1 y \to \boldsymbol{AF} L(y)\big) \wedge \\
 & & \qquad\qquad \forall y \big(\boldsymbol{AF} L(y) \to \exists x (x \preccurlyeq y \wedge L(x))\big)
\end{array}
$$

where $x \preccurlyeq y$ is an abbreviation for $(x \prec y \vee x \approx y)$. Finally, let

$$
\begin{array}{rcl}
A'' & = & A_1 \wedge \ldots \wedge A_5 \wedge A''_6 \wedge A''_7 \wedge A_8 \wedge A''_9; \\
B'' & = & \exists x \big(L(x) \wedge \boldsymbol{EG}(A'' \wedge \neg \exists y (y \prec x))\big); \\
C'' & = & \forall x \boldsymbol{AF} L(x).
\end{array}
$$

Proposition 6. *The formula $B'' \wedge C''$ is* **QCTL***-satisfiable.*

Proof. Let $\mathfrak{M}_0$ be the model defined on page 6. We extend I_0 on p so that

$$(\mathfrak{M}_0, w_i) \models p \iff i = 2k, \text{ for some } k \in \mathbb{N},$$

i. e., we put p to be true exactly in 'even' worlds.

Then $(\mathfrak{M}_0, w_0) \models B'' \wedge C''$; corresponding check is left to the reader. $\square$

Let us use again the notations introduced in Section 6.

Let $\mathfrak{M} = \langle W, R, D, I \rangle$ be a model and w^* be a world in it such that $(\mathfrak{M}, w^*) \models B'' \wedge C''$.

Since $(\mathfrak{M}, w^*) \models B''$, there are $a_0 \in D(w^*)$ and a path π starting in w^* such that

- $L^{\pi_0}(a_0)$ is true;
- $b \prec^{\pi_k} a_0$ does not true, for any $k \in \mathbb{N}$ and for any $b \in D(\pi_k)$;
- $(\mathfrak{M}, \pi_k) \models A''$, for any $k \in \mathbb{N}$.

LEMMA 11. *The relation $\approx^{\pi_k}$ is a congruence with respect to the relation $\prec^{\pi_k}$, for any $k \in \mathbb{N}$.*

PROOF immediately follows from $(\mathfrak{M}, \pi_k) \models A_1 \wedge A_2$. $\square$

Because of Lemma 11, we again may define congruence classes but just for worlds in the path π.

LEMMA 12. *The relation $\prec^{\pi_k}$ is a strict linear order on D^{π_k} and $\prec_1^{\pi_k}$ is the successor relation on D^{π_k} with respect to $\prec^{\pi_k}$, for any $k \in \mathbb{N}$.*

PROOF immediately follows from $(\mathfrak{M}, \pi_k) \models A_3 \wedge A_4$. $\square$

From the condition $(\mathfrak{M}, \pi_0) \models A_5$ it follows that there are $a_1, a_2, a_3, \ldots \in D(\pi_0)$ such that

$$a_0 \prec^{\pi_0} a_1 \prec^{\pi_0} a_2 \prec^{\pi_0} a_3 \prec^{\pi_0} \ldots$$

and equivalence classes generated by $a_0, a_1, a_2, a_3, \ldots$ are pairwise different.

LEMMA 13. *For any $k \in \mathbb{N}$ and for any $a, b \in D(\pi_0)$,*

$$\begin{array}{lcl} a \prec_1^{\pi_0} b & \Longrightarrow & a \prec_1^{\pi_k} b; \\ a \not\approx^{\pi_0} b & \Longrightarrow & a \not\approx^{\pi_k} b. \end{array}$$

PROOF. Let $a \prec_1^{\pi_0} b$, for some $a, b \in D(\pi_0)$. We prove that $a \prec_1^{\pi_k} b$ by induction on k. If $k = 0$ then the statement is trivial.

Let $a \prec_1^{\pi_k} b$ and let α be an interpretation of variables in $D(\pi_k)$ such that $\alpha(x) = a$, $\alpha(y) = b$. There are two possible cases: $(\mathfrak{M}, \pi_k) \models^\alpha p$ and $(\mathfrak{M}, \pi_k) \not\models^\alpha p$.

Case $(\mathfrak{M}, \pi_k) \models^\alpha p$. Suppose $a \not\prec_1^{\pi_{k+1}} b$. Since $(\mathfrak{M}, \pi_{k+1}) \models A_6$, we have $(\mathfrak{M}, \pi_{k+1}) \not\models^\alpha \boldsymbol{AF}(x \prec y \wedge \neg p)$. This means that there is a path σ starting in π_{k+1} such that $(\mathfrak{M}, \sigma_m) \not\models^\alpha (x \prec y \wedge \neg p)$, for any $m \in \mathbb{N}$. We define the path τ: $\tau_0 = \pi_k$ and $\tau_{n+1} = \sigma_n$, for any $n \in \mathbb{N}$; i.e., τ starts in π_k and then goes along σ. Since $(\mathfrak{M}, \tau_0) \models^\alpha p$, we have $(\mathfrak{M}, \tau_0) \not\models^\alpha (x \prec y \wedge \neg p)$ and from the same condition for all states in σ we obtain $(\mathfrak{M}, \tau_n) \not\models^\alpha (x \prec y \wedge \neg p)$, for any $n \in \mathbb{N}$. But then $(\mathfrak{M}, \tau_0) \not\models^\alpha \boldsymbol{AF}(x \prec y \wedge \neg p)$, i.e., $(\mathfrak{M}, \pi_k) \not\models^\alpha \boldsymbol{AF}(x \prec y \wedge \neg p)$. Then, since $(\mathfrak{M}, \pi_k) \models A_6$, we have $(\mathfrak{M}, \pi_k) \not\models^\alpha x \prec y$, i.e., $a \not\prec_1^{\pi_k} b$. But, by induction hypothesis, it is not the case, hence the assumption $a \not\prec_1^{\pi_{k+1}} b$ is not true, and we have $a \prec_1^{\pi_{k+1}} b$.

Case $(\mathfrak{M}, \pi_k) \not\models^\alpha p$. We have $(\mathfrak{M}, \pi_k) \models^\alpha \neg p$ and with the same argumentation (using p instead of $\neg p$ and vice versa) we again obtain $a \prec_1^{\pi_{k+1}} b$.

Let $a \approx_1^{\pi_0} b$, for some $a, b \in D(\pi_0)$. The proof that $a \prec_1^{\pi_k} b$ proceeds by induction on k in the same way (with use of A_7 instead of A_6); we left the details to the reader. □

Lemma 14. *Let $k \in \mathbb{N}$. Then there is $m \in \{0, \dots, k\}$ such that $L^{\pi_k}(a_m)$ is true.*

Proof proceeds by induction on k.

Let $k = 0$. Then we must prove that $L^{\pi_0}(a_0)$ is true; but we have $L^{\pi_0}(a_0)$ to be true by choosing of a_0.

Let the statement be true for k; we prove it for $k + 1$. By induction hypothesis, we have $L^{\pi_k}(a_m)$, for some $m \in \{0, \dots, k\}$. Let α be an interpretation of variables in π_k such that $\alpha(x) = a_m$, $\alpha(y) = a_{m+1}$. By Lemma 13, we have $a_m \prec_1^{\pi_k} a_{m+1}$ and hence $(\mathfrak{M}, \pi_k) \models^\alpha \boldsymbol{AF}L(y)$ because $(\mathfrak{M}, \pi_k) \models A_9''$.

Suppose $L^{\pi_{k+1}}(a_0), \dots, L^{\pi_{k+1}}(a_{k+1})$ are not true. Then, in particular, $L^{\pi_{k+1}}(a_0), \dots, L^{\pi_{k+1}}(a_{m+1})$ are not true and hence $(\mathfrak{M}, \pi_{k+1}) \not\models^\alpha \exists x(x \preccurlyeq y \wedge L(x))$. Since $(\mathfrak{M}, \pi_{k+1}) \models A_9''$, we obtain $(\mathfrak{M}, \pi_{k+1}) \not\models^\alpha \boldsymbol{AF}L(y)$. This means that there is a path σ starting in π_{k+1} such that $(\mathfrak{M}, \sigma_l) \not\models^\alpha L(y)$, for any $l \in \mathbb{N}$.

We define the path τ: $\tau_0 = \pi_k$ and $\tau_{n+1} = \sigma_n$, for any $n \in \mathbb{N}$.

We claim $(\mathfrak{M}, \tau_n) \not\models^\alpha L(y)$, for any $n \in \mathbb{N}$. Indeed, it is the case for any $n \in \mathbb{N}^+$ by the choice of σ. As for $n = 0$, we have $\tau_0 = \pi_k$ and hence $(\mathfrak{M}, \tau_0) \models^\alpha L(x)$. Supposing $(\mathfrak{M}, \tau_0) \models^\alpha L(y)$, we obtain $(\mathfrak{M}, \tau_n) \models^\alpha x \approx y$ because $(\mathfrak{M}, \tau_0) \models A_8$. But then

$a_m \approx^{\pi_k} a_{m+1}$. From Lemmas 11 and 12 we obtain $a_m \not\prec^{\pi_k} a_{m+1}$, in particular, $a_m \not\prec_1^{\pi_k} a_{m+1}$, but this is impossible by Lemma 13. Thus $(\mathfrak{M}, \tau_0) \not\models^\alpha L(y)$.

Since $(\mathfrak{M}, \tau_n) \not\models^\alpha L(y)$ holds for any $n \in \mathbb{N}$, we obtain $(\mathfrak{M}, \tau_0) \not\models^\alpha \boldsymbol{AF}L(y)$, i.e., $(\mathfrak{M}, \pi_k) \not\models^\alpha \boldsymbol{AF}L(y)$, but this is not so. Hence our assumption is not true, therefore at least one of $L^{\pi_{k+1}}(a_0), \ldots, L^{\pi_{k+1}}(a_{k+1})$ is true. □

Now we are ready to prove a lemma that is similar to Lemmas 6 and 9. Recall that $\pi_0 = w^*$.

Lemma 15. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_m$, for some $m \in \mathbb{N}$.*

Proof. Since $(\mathfrak{M}, w^*) \models C$, in any path starting in w^* there is a world w such that $L^w(b)$ is true. In particular, this is true for the path π. Hence $L^{\pi_k}(b)$ is true, for some $k \in \mathbb{N}$. Then, by Lemma 15, there is $m \in \{0, \ldots, k\}$ such that $L^{\pi_k}(a_m)$ is true. Since $(\mathfrak{M}, \pi_k) \models A_8$, we have $b \approx^{\pi_k} a_m$. By Lemma 13, we obtain $b \approx^{\pi_0} a_m$, i.e., $b \approx^{w^*} a_m$. □

Proposition 7. *The structures $\langle D^{w^*}, \prec^{w^*} \rangle$ and $\langle \mathbb{N}, < \rangle$ are isomorphic.*

Proof. Follow to the proof of Proposition 3. □

Let φ be some closed classical first-order formula. We define $Emb''(\varphi)$:

$$Emb'(\varphi) \quad = \quad B'' \wedge C'' \wedge Congr(\varphi) \to T(\varphi).$$

Lemma 16. $\varphi \in \mathbf{QCL}_{fin} \iff Emb''(\varphi) \in \mathbf{QCTL}$.

Proof proceeds with the same argumentation as the proof of Lemma 7; the difference is in use Proposition 7 instead of Proposition 3. □

Corollary 4. *The logic $\mathbf{QCTL} \restriction \{\boldsymbol{AF}\}$ is not recursively enumerable.*

9 Corollaries

First of all, now we are able to give a proof for Theorem 1. Indeed, let M be a set of modalities as in Theorem 1. If $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ or $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then the logic **QCTL** $\restriction M$ is recursively enumerable and even finitely axiomatizable by Proposition 1. Suppose $M \not\subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ and $M \not\subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$. Then at least one of the following conditions holds:

- M contains $\boldsymbol{AX}$ or $\boldsymbol{EX}$ and M contains $\boldsymbol{AG}$ or $\boldsymbol{EF}$;
- M contains $\boldsymbol{EU}$;
- M contains $\boldsymbol{AU}$, or $\boldsymbol{AF}$, or $\boldsymbol{EG}$.

In every of these cases the logic **QCTL** $\restriction M$ is not recursively enumerable by Corollaries 2, 3, and 4.

Now we give some other corollaries of Theorem 1 and its proof. Below let $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EG}, \boldsymbol{AF}, \boldsymbol{EF}, \boldsymbol{AU}, \boldsymbol{EU}\}$.

9.1 Finite axiomatizability

Note that any recursively enumerable fragment we consider here is also finitely axiomatizable; thus we have the following statement.

COROLLARY 5. *The logic* **QCTL** $\restriction M$ *is finitely axiomatizable if and only if* $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ *or* $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$.

We make a remark. From Theorem 1 and Corollary 5 it follows that a fragment **QCTL** $\restriction M$ is recursively enumerable if and only if it is finitely axiomatizable. In general, recursive enumerability of a logic is not equivalent to its finite axiomatizability; it is equivalent just to its recursive axiomatizability. In our case such criterion is possible maybe because of finite number of fragments we consider.

Now let us turn to the other side of axiomatizability. We mean completeness.

9.2 Kripke completeness

Recall that a logic L is called *Kripke complete* if there is a class $\mathfrak{C}$ of (predicate) Kripke frames such that $L = \{\varphi : \mathfrak{C} \models \varphi\}$ where $\mathfrak{C} \models \varphi$ means φ is true in any frame in $\mathfrak{C}$.

Any fragment $\mathbf{QCTL} \restriction M$ is Kripke complete by its definition; so, the question about Kripke completeness makes sense just for logics defined in other ways. Here we concern calculi.

We specify what we understand under a calculus. We assume a calculus to be defined by means of two sets: a set $\mathcal{A}$ of axioms and a set $\mathcal{R}$ of inference rules. These sets assumed to be recursively enumerable, moreover any inference rule assumed to be effective, i. e., as an algorithm computing the resulting formula. Of course, finite axiomatizations provides us with calculi.

Note that the set of all derivable formulas in a calculus (in our understanding) is recursively enumerable: to get an algorithm enumerating derivable formulas we may use an algorithm constructing a consequence of all derivations. The last algorithm exists because of algorithmic conditions we claim for the sets of axioms and inference rules.

Let $\mathcal{R}$ be a set of inference rules, X be a set of formulas. We denote by $\mathcal{C}_{\mathcal{R}}(X)$ the least set of formulas containing X and closed under inference rules in $\mathcal{R}$, generalization, modus ponens, and substitution (in an appropriate language). Let also $X \oplus_{\mathcal{R}} Y = \mathcal{C}_{\mathcal{R}}(X \cup Y)$.

Let L be a logic (a set of formulas). Let us call a set $\mathcal{R}$ of inference rules *L-admissible* if $\mathcal{C}_{\mathcal{R}}(L) \subseteq L$.

Let us call a set M of modalities *quite expressive* if the modalities in at least one of the sets $\{\boldsymbol{AX}, \boldsymbol{AG}\}$, $\{\boldsymbol{EU}\}$, $\{\boldsymbol{AF}\}$ can be expressed via the ones in M (maybe with use of the connectives).

COROLLARY 6. *Let M be a quite expressive set of modalities, S be a recursively enumerable set of propositional formulas in the language of* $\mathbf{CTL} \restriction M$ *such that* $S \subseteq \mathbf{CTL} \restriction M$*, and $\mathcal{R}$ be a* $\mathbf{QCTL}$*-admissible set of inference rules. Then the logic* $\mathbf{QCL} \oplus_{\mathcal{R}} S$ *is not Kripke complete.*

PROOF. We give just a sketch of a proof. Let us use the denotation $QS = \mathbf{QCL} \oplus_{\mathcal{R}} S$. The logic QS may be viewed as a calculus with the axiom set $\mathbf{QCL} \cup S$, therefore it is recursively enumerable. Suppose it is Kripke complete. Then there is a class $\mathfrak{C}$ of Kripke frames such that $QS = \{\varphi : \mathfrak{C} \models \varphi\}$.

First of all observe that (modulo modality equivalence) at least one of the formulas $A \wedge B \wedge C$, $A' \wedge B \wedge C$, $B'' \wedge C''$ is a formula in the language of QS; this is so because M is quite expressive. Let us denote it by Φ. We claim

$$\varphi \in \mathbf{QCL}_{fin} \iff \Phi \wedge Congr(\varphi) \to T(\varphi) \in QS,$$

for any closed classical first-order formula φ.

Suppose $\varphi \notin \mathbf{QCL}_{fin}$. Then there is a classical model $\mathfrak{S} = \langle S, J\rangle$ such that S is finite and $\mathfrak{S} \not\models \varphi$; we may assume $S = \{0, \ldots, n\}$, for some $n \in \mathbb{N}$.

Observe that $\neg\Phi \notin QS$. Indeed, it follows from Propositions 2, 4, and 6 that $\neg\Phi \notin \mathbf{QCTL}$. But $\mathbf{QCL} \subset \mathbf{QCTL}$ and $S \subseteq \mathbf{CTL} \upharpoonright M \subset \mathbf{QCTL}$, therefore $QS \subseteq \mathbf{QCTL}$ because $\mathcal{R}$ is $\mathbf{QCTL}$-admissible.

Since $\neg\Phi \notin QS$, there is a frame $\mathfrak{F}(D) = \langle W, R, D\rangle$ such that $\mathfrak{F}(D) \in \mathfrak{C}$ (or $\mathfrak{F} \in \mathfrak{C}$ where $\mathfrak{F} = \langle W, R\rangle$) and $\mathfrak{F}(D) \not\models \neg\Phi$. Then there is a model $\mathfrak{M} = \langle W, R, D, I\rangle$ and a world $w^* \in W$ such that $(\mathfrak{M}, w^*) \not\models \neg\Phi$, i. e., $(\mathfrak{M}, w^*) \models \Phi$.

It follows from Propositions 6, 5, and 7 that the structures $\langle D^{w^*}, \prec^{w^*}\rangle$ and $\langle \mathbb{N}, <\rangle$ are isomorphic, i. e., there is an isomorphism $f : \mathbb{N} \to D^{w^*}$ for the structures. Let us consider the model $\mathfrak{M}' = \langle W, R, D, I'\rangle$ where I' is defined as I with the only difference for predicate letters occurring in φ: if P is m-ary letter occurring in φ and $b_1, \ldots, b_m \in D(w^*)$ then we put

$$\langle b_1, \ldots, b_m\rangle \in I'(w^*, P) \;\leftrightharpoons\; \text{there are } k_1, \ldots, k_m \in S \text{ such that } \langle k_1, \ldots, k_m\rangle \in J(P) \text{ and } b_i \in f(k_i), \text{ for any } i \in \{1, \ldots, m\}.$$

Then we obtain $(\mathfrak{M}', w^*) \not\models \Phi \wedge Congr(\varphi) \to T(\varphi)$, and hence $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin QS$; the details are left to the reader.

Now suppose $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin QS$. Since $QS \subseteq \mathbf{QCTL}$, we have $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin \mathbf{QCTL}$, and hence $\varphi \notin \mathbf{QCL}_{fin}$ by Lemmas 6, 9, and 15.

But then QS is not recursively enumerable, and we have a contradiction. Hence QS is not Kripke complete. □

9.3 The case of constant domains

Kripke frame $\mathfrak{F}(D) = \langle W, R, D \rangle$ is said to be a frame with *constant domains* if

$$wRw' \implies D(w) = D(w'),$$

for any $w, w' \in W$. The constant domain condition means that new elements do not appear when we go from one state to another, i. e., any element we may deal with in some future state is available in the current state, too.

Let $\mathbf{QCTL}^{cd}$ be a logic of serial Kripke frames with constant domains. Note that the propositional fragment of $\mathbf{QCTL}^{cd}$ is the logic **CTL**, i. e., $\mathbf{QCTL}^{cd}$ as well as **QCTL** is a conservative first-order extension of **CTL**. Here we put and answer the following question: do the presented results remain to be true if we replace **QCTL** with $\mathbf{QCTL}^{cd}$? And the answer is 'yes, of course'.

Indeed, it is sufficient to observe that the formulas $A \wedge B \wedge C$, $A' \wedge B \wedge C$, and $B'' \wedge C''$ are satisfiable in some models based on a serial frame with constant domains; see the proofs of Propositions 2, 4, and 6.

9.4 More simple fragments

We make remarks about fragments with restrictions on predicate letters. First of all, observe that it is possible to define $\approx$ and $\prec$ via L. So, for example in w^* (see Section 6)

$$\begin{array}{lll} x \approx y & \text{means} & \boldsymbol{AG}(L(x) \leftrightarrow L(y)); \\ x \prec y & \text{means} & \boldsymbol{AF}(L(x) \wedge \neg L(y) \wedge \boldsymbol{AF}L(y)). \end{array}$$

As for the $\mathbf{QCL}_{fin}$, its fragment with a binary predicate letter P is not recursively enumerable. It is known, see [8], that a binary predicate letter may be simulated with two unary letters: for example $P(x,y)$ may be simulated with $\boldsymbol{EX}(P'(x) \wedge P''(y))$. This means that three unary letters are sufficient to prove that the correspondent fragment of **QCTL** is not recursively enumerable.

Moreover even two unary letters are enough. It is known that the theory of finite models with symmetric irreflexive binary relation is not recursively enumerable; see [9]. But if a letter P corresponds to such relation then $P(x,y)$ may be simulated with $\boldsymbol{EX}((P'(x) \wedge \neg P'(y)) \vee (\neg P'(x) \wedge P'(y)))$.

As for number of individual variables, it seems to be truthful that three ones are enough. But in view of [7], we think that even two variables are enough.

10 Conclusion remarks

Let M be some quite expressive set of modalities. Suppose we are asked: why the fragment $\mathbf{QCTL} \upharpoonright M$ is not recursively enumerable? As a possible answer we may say that this is because one may simulate positive integers using the language of the fragment. But why one may do this?

To answer the question let us turn to the notion of path. A path π in a frame $\mathfrak{F} = \langle W, R \rangle$ is an infinite consequence of worlds $\pi_0, \pi_1, \pi_2, \ldots$ with the condition $w_k R w_{k+1}$, for any $k \in \mathbb{N}$. This means that any path is a map from $\mathbb{N}$ into W. Thus, we have positive integers as paths. Some modalities allow us to 'catch' them because of their definition in Kripke models.

But it is not the case for the pairs $\boldsymbol{AX}, \boldsymbol{EX}$ and $\boldsymbol{AG}, \boldsymbol{EF}$. This is so because these modalities are quite 'simple': accessibility relations for them are first-order definable. For example, for $\boldsymbol{AX}$ we need just seriality of R. The modality $\boldsymbol{AG}$ seems to be more complicated. It corresponds to the reflexive and transitive closure of R which is not first-order definable via R. And indeed, if we use $\boldsymbol{AX}$ and $\boldsymbol{AG}$ simultaneously then we are able to 'catch' positive integers. But if we use $\boldsymbol{AG}$ only then we 'lose' R and, in fact, we have just some reflexive and transitive accessibility relation.

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Deontic ‘cocktail’ according to E. Mally’s receipt[1]

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abstract. In 1926, Ernst Mally, an Austrian logician, has introduced a system of deontic logic in which he has proposed three fundamental distinctions which proved to be important in the context of the further development of the logic of norms. It is argued that in his philosophical considerations Mally has introduced a number of important distinctions concerning the very concept of norm, but by getting them confused in introducing the subsequent formalisms he failed to formally preserve them. In some of his philosophically made distinctions Mally apparently foresaw contemporary trends in logic of norms. To some extent this particular feature of Mally’s system open wide opportunities to reconstruct — with the corresponding renovations — his ill-formed Deontik into many nowadays known systems of logic of norms and thus provides a fertile ground for this kind of research.

Keywords: deontic logic, Mally, agency, ought, obligation

1 Introduction

Conceptual considerations about developing a special kind of logic capable to model norms and reasoning about norms date back at least to the Middle Ages, but it was not until von Wright’s deontic systems have been introduced in 1951 that the first a viable and sound system of deontic logic was proposed. Standard von Wright-type deontic logic which was thus launched has developed into one of the most significant trends in the area of logic of norms, although not without being criticized [2], [3]. Among those criticisms, the most troublesome seemed to be the so called paradoxes of absolute deontic

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logic that targeted one of the central concepts of standard deontic logic, namely the concept of deontically perfect state of affairs, the one in which all norms are assumed to be fulfilled [4, p. 401]. Today, when diverse trends in this area have emerged, these paradoxes are regarded to be characteristic rather of the type of deontic system, than of the logic of norms as a part of contemporary logic [7, p. 172].

As a logical investigation of normative reasoning, or reasoning in the framework of normative systems, logic of norms is constantly moving towards more adequate norms understanding which is obtained in the framework of their formal representation. In the very beginning of its development, the goal of norms' formal representation was thought to be achieved by means of deontic calculi, that later have been supplemented by standard semantic structures, which yet later has proved to be in many aspects inadequate [5, p. 148]. Contemporary normative formalisms tend to regard semantic aspects of norms as more fundamental then inferential relations among them.

Both the structure of norm and as well as the relations established between the elements of it are assumed as relevant for being modeled by means of a logical theory. For this reason, they form one of the key issues in defining norm in the sense of logic. Norms consist of four basic elements [4, p. 380]:

1) actions, or states of affairs, which can be;
2) (according to norm character) allowed, prescribed or prohibited from being performed by agents;
3) agents as subjects of norms;
4) conditions (actions or states of affairs) for norms' emerging or ceasing.

Deontic logic proper regards (1) and (2) as more fundamental for logical investigations about norms and for this reason it may be called 'objectivist' trend in the logic of norms. Deontic logic pursues logical aspects of the relations between (1) and (2), on the one hand, and norms' obedience or disobedience, on the other. This 'objectivist' approach relies to a large extent on an assumption that proper understood norms are impersonal timeless absolute (condition-free) regulations. In this way, norms, when they are obeyed, generate a normatively ideal (perfect) state of affairs, or deontically ideal world. According to the deontic approach, norms proper may enter conditional regulations becoming thus the elements of them together with (3), (4), temporal, epistemic and other modalities, but this fact does not preclude deontic theories from studying their central concept of normative relation between (1)–(2) and deontically perfect worlds.[2]

Indeterministic agent-dependent logic of norms regards issues (3)–(4) as more significant than (1)–(2) and is a major contemporary rival of deontic logic. Norms' analysts belonging to this trend see the relation between agent, (3), and its strategy in goal-oriented activities as key issue for studying human beings' normative behavior in which deontically understood norms (1)–(2) form a correlative element of agent's strategy. Indeterministic logic of norms stems out of von Wright's ideas of logic of action [3] and A. Prior's theory of branching time [17], [18] and is being developed in the works of J. Horty [11], N. Belnap [9] and others.

Indeterministic logical theories of obligation introduce special *stit*-operator of agency which may be understood in different ways depending on the interpretations of concepts of history, time and moment [8], all of them are related to agent's actions. These theories incorporate two important ideas concerning logical analysis of agency. The first is that of agent's ability to do something as closely related to both what an agent ought to do and what should have done. J. Horty calls it Meinong\Chisholm analysis and reports that it can be traced back to the works of some German and Austrian philosophers [11, pp. 44–46]. The other one explicitly marks the borderline distinction between SDL altogether with its further developments and indeterministic theories and proposes the concepts of branching time and corresponding linear histories as intrinsic to agent's behavior in such a way that the latter is assumed to be dependent of agent's previously made choices in the way that it secures its freedom in what concerns its future choices. Contrary to that, SDL as well as its contemporary versions rely on certain determined

[2]See [7] for a substantial exposition of deontic logic proper.

future state of affairs thus leaving no room for agent's future choices otherwise than being caused by those previously made.

In this paper, some arguments are proposed to support of the idea that Austrian logician Ernst Mally should be added to the list of those German-speaking philosophers whose conceptual considerations of the issue gave rise to the idea of agent-dependent normativity. It is also suggested that in his *Deontik* Mally was the first to introduce a deontically understood agential ought as distinct from agent-free impersonal obligation, though, apparently, did it in somewhat vague way.

2 Who is Mally?

Ernst Mally (1879–1944), an Austrian logician, a pupil of Alexius Meinong and the author of several philosophical writings, was born in Slovenia which then has been a part of Austro-Hungarian Empire. In 1926 Mally published a book *Grundgesetze des Sollens: Elemente der Logik des Willens* ('The Basic Laws of Ought: Elements of the Logic of Willing') [16] in which he proposed a logical theory which happened to become the first approach to formulate a system of deontic logic. Ernst Mally called his theory *Deontik* and thus became the author of the both, the first system of deontic logic and the term for this branch of logic.

As a viable formalism, his system has proved to be unsuccessful. Mally's book is almost 100 pages long but the chapter in which the calculi is proposed is hardly longer than 15 pages and then throughout other 15 pages 35 theorems are given followed by concise explanations. In the rest of his book, Chapters III and IV, he pursues the surprising consequences his system yields. In doing so he notoriously tries to show that the reason for those strange consequences to follow has to be looked for not in the axiomatic basis of the system, but instead in the properties of logic of obligation and will itself, or in the idea of pure ethics which he advocates in his book.

E. Mally held very radical national socialist views during the whole of his life. He was a teenager when he has joined one of Austrian radical movements, later he became a member of an Austrian radical society that have been supporting the idea of the Anschluss of Austria to Germany even before the World War I, and joined NSDAP immediately after it had happenned. He was an active Nazi-party member and in most of his papers written during the last decade of his life he argues for the Nazi ideology [22].

Yet despite these 'hard' facts of his biography his philosophical and logical heritage never fully went into oblivion as one may well be inclined to think. His intellectual life is usually divided into 3 stages. The first stage was devoted to the object theory [21], [6]. During the second Mally studied various philosophical issues sometimes with the help of the object theory, and the third saw his politically oriented writings. The book in which his Deontik is proposed belongs to the second period and was written before he has started to pursue his political activities also in the philosophical papers. Despite the fact that his system has turned out to be ill-formed, and, perhaps even because of it, Mally's deontic logic remained being mentioned whenever the issue of the starting point of the development in this area of deontic logic has been touched. However, in most cases the mentioning is being done in the sense of unsuccessful start.

Unlike his teacher, A. Meinong, Ernst Mally has founded no philosophical school, yet he had several pupils. One of them, Karl Wolf, in collaboration with his pupil, Paul Weingartner, in 1971 prepared and published the modern edition of E. Mally's *Grundgesetze des Sollens: Elemente der Logik des Willens* [16] with the substantial philosophical foreword.

The company of Mally's critics includes many outstanding logicians that have essentially contributed to the field of logic of norms: K. Menger, who was the first to attack Deontik, G.von Wright [2], [3]; D. Follesdal and R. Hilpinen [10], J. Wolensky [20]. O. Weinberger [19] suggests an outline of Mally's system; J.-G. Lokhorst proposed several reconstructions of Mally's *Deontik* along with some critical renovations [12], [13], [14], [15].

According to most of them, there are three main reasons responsible for Mally's logical failure. As Lokhorst summarizes them in [12] these are (1) the classical two-valued propositional calculus which forms the non-deontic part of Mally's system and (2) the fact that some deontic axioms are vague and need modifications, and (3) both (1)–(2). (1) is the issue particularly criticized by K. Menger, and is also touched upon by Lokhorst, too:

If Mally's deontic principles are added to a system in which the so-called paradoxes of material and strict implication are avoided, many of the 'surprising' theorems (such as (34) and (35)) are no longer derivable and $A \leftrightarrow\, !A$ is no longer derivable either. But most of the theorems which Mally regarded as 'plausible' are still derivable. The resulting system is closely related to Anderson's relevant deontic logic [12].

3 Mally and Jorgensen's dilemma

Throughout his book Mally neither explicitly specifies the non-deontic part of his system, nor he accepts any propositional tautologies as belonging to his system. Thus, it would be unfair to maintain as Lokhorst does [12] that Mally proposes a (classical) propositional basis for his system in the way von Wright or later deontic logicians did and that has become quite standard in the second half of XX century [5].

Instead, we find a number of philosophical explanations concerning the nature of implication from which one may conclude that he clearly distinguishes his system as the logic of what is thinkable (*Denklogik*) or the object logic (*Gegenstandlogik*) from the system suggested in *Principia Mathematica* by Russell and Whitehead.[3] He holds the view that the latter describes the logical relations between propositions understood as propositional functions and calls it logistics (*Logistik*) [16, p. 236, 320 (notes 4–6)]. Mally purports to make this distinction as sharp as possible, especially in the chapters III and IV of his book where he notoriously advocates his ill-formed system. In doing so he believes that willing and obligatoriness are conceptual objects that have their special logic which is different from what he calls logistic [16, p. 237]. Unfortunately, whereas in his philosophical explanations Mally indeed draws this distinction between the two kinds of logic, in his formalism he gets them confused. This is one of several confusions made in the *Deontik* that apparently led to the failure of the system.

Another confusion in his book follows immediately out of the one just discussed. In the same way as with the kinds of logic, Mally says that a rigor distinction should be made between the

[3]See also [6] on this point.

kinds of implications that hold in case of propositions and in case of objects, or states of affairs (*Sachverhalt*), respectively. A *impliziert* B (A implies B)[4] and A *fordert* B (A demands B)[5] are distinct from each other and are meant to be propositional and normative respectively.[6] The former is apparently truth-functional and close to what one may call material implication, whereas the latter looks more like formal implication [16, note 31]. Consequently, in what concerns (1) Mally did go wrong but not in the way diagnosed by K. Menger or J.-G. Lokhorst, but rather in the other way round. He has put the two distinct types of inferential relations into one system and thereby has got different ontological assumptions confused, and he did so by applying the truth-functional propositional patterns of logical inferences to propositions he himself takes in different, sometimes prescriptive, sense.

It seems that in these wrongly understood inferential relations among propositions expressing norms Mally indeed had been the first to overlook the problem [2, p. 291] which later has been called Joergensen's dilemma. Norms' analysts widely recognize this dilemma which amounts to the following. The practice of defining logical consequence in terms of satisfiability rests on the assumption

[4]The relation of implication is a relation between propositions describing two states of affairs, A and B respectively, that are understood or take place in such a way that A implies B, but this relation is not the one to be found between propositions expressing what is being thought or willed or ought to be, explains E.Mally before he turns to outline his system. From the fact that the state of affairs A does not happen any other state of affairs follow and an actual state of affairs is implied by whatsoever state of affairs [16, pp. 238–240].

[5]'When something is being desired, everything in absence of which this volition may not realize, is being also desired. This is the essence of the volition.' [16, p. 246]. 'It lies in the very essence of willing that willing is just willing whatsoever that willing implies. . . It has happened to everyone that in some unforeseen circumstances in which one finds oneself to be obliged to apologize for one's undesired behavior it is natural to say that one did not know the consequences if his or her actions, but should have thought that this undesired thing would happen' [16, p. 273].

[6]The fact that these are distinct is obvious in his definition of connective f: $A \text{ f } B = A \to !B$, and in Axiom III: $(A \text{ f } B) \leftrightarrow !(A \to B)$. With the help of these Mally suggests the way how they can be expressed in terms of each other. It is also clear that he sees the right parts of the two as equipollent, taking so far their equipollency as innocuous for his system.

that truth values are necessary properties of propositions describing states of affairs. Contrary to descriptive propositions, prescriptively taken norms lack truth value. Consequently, either no logic of norms proper is possible, or the idea of truth value based on the concept of satisfiability of propositions needs reconsideration. Most results in logic of norms so far have been acquired in the framework of the latter line.[7]

4 Willing, obligatoriness and norms in Mally's system

What concerns (2) Lokhorst suggests three sound reformulations of Mally's system and does so by proposing alternative non-deontic bases that enrich Mally's authentic system with additional postulates as well as by modifying some of Mally's original axioms. Lokhorst's reconstructions result respectively in two versions of RD, a relevant deontic system with the system R as its non-deontic part [12] and RD with a propositional constant of andersonian type [15], and KD, a version of standard von-Wright-type deontic system [12].

Mally has in mind three different concepts of what obligatoriness may mean when applied to a state of affairs, to a conceptual (intentional) object and to a relation among them respectively. Whereas the first and the third may be called norms in some sense, the second definitely may not, for it is meant to express Mally's philosophical idea of rationally put human will which in order to be feasible should be understood as a conceptual objec t and, consequently, as a logically consistent object. Mally suggests his system for the sake of showing how these distinct concepts logically relate to each other, and obviously fails on this point because gets the three formally confused.

Mally is aware of the fact that the three are distinct and holds the view that each of them requires different logic. The idea that the relations between norms and the propositions expressing the conditions for them as well as norms' obedience or disobedience are non-truth-functional is plain in what Mally says when trying to

[7]For a survey of the development of logic of norms see [4]. [5] suggests an outline of the development of the concept of norm in the framework of logic of norms.

justify the 'strange' consequences' his system yields. He is also very accurate in distinguishing the types of implications, namely, formal and material, especially when speaking of logical dependencies between the three.

5 Mally's *Deontik*

The non-deontic part of Mally's system consists of the sentential letters A, B, C, P and Q; the sentential variables M and N (these two groups of symbols refer to states of affairs); the sentential constants V (the *Verum*, Truth) and Λ (the *Falsum*, Falsity); the propositional quantifiers $\exists$ and $\forall$, and the connectives $\neg$, &, $\vee$, $\rightarrow$ and $\leftrightarrow$. Λ is defined by $\Lambda = \neg$ V.

The *deontic* part of Mally's vocabulary includes the unary connective !, the binary connectives f and ∞, and the sentential constants $\cup$ and $\cap$. He supplies the *deontic* part of his system with the following definitions:

Def. f . A f B = A $\rightarrow$!B.

Def. ∞. A ∞ B = (A f B) & (B f A)

Def. $\cap$. $\cap = \neg$ U

There are five axioms in Mally's system. They are given in the Table together with original Mally's symbolisms and Lokhorst's formalizations of them.

	Basic principles	Mally's formalization	Lokhorst's formalization
i	*If A requires B and if B then C, then A requires C.*	((A f B) & (B $\rightarrow$ C)) $\rightarrow$ (A f C)	((A $\rightarrow$!B) & (B $\rightarrow$ C)) $\rightarrow$ (A $\rightarrow$!C)
ii	*If A requires B and if A requires C, then A requires B and C.*	((A f B) & (A f C)) $\rightarrow$ (A f (B & C))	((A $\rightarrow$!B) & (A $\rightarrow$!C)) $\rightarrow$ (A $\rightarrow$!(B & C))
iii	*A requires B if and only if it is obligatory that if A then B.*	(A f B) $\leftrightarrow$!(A $\rightarrow$ B)	(A $\rightarrow$!B) $\leftrightarrow$!(A $\rightarrow$ B)
iv	*There is an unconditionally obligatory which is obligatory.*	$\exists$U !U	!U

v	*The unconditionally obligatory does not require its own negation.*	¬(U f ∩)	¬(U → !∩)

Mally derived the following theorems from his axioms [16, pp. 252–269].

(1) (A f B) → (A f V)

(2) (A f Λ) ↔ ∀M (A f M)

(3) ((M f A) ∨ (M f B)) → (M f (A ∨ B))

(4) ((M f A) & (N f B)) → ((M & N) f (A & B))

(5) !P ↔ ∀M (M f P)

(6) (!P & (P → Q)) → !Q

(7) !P → !V

(8) ((A f B) & (B f C)) → (A f C)

(9) (!P & (P f Q)) → !Q

(10) (!A & !B) ↔ !(A & B)

(11) (A ∞ B) ↔ !(A ↔ B)

(12) (A f B) ↔ (A → !B) ↔ !(A → B) ↔ !¬(A & ¬B) ↔ !(¬A ∨ B)

(13) (A → !B) ↔ ¬(A & ¬!B) ↔ (¬A ∨ !B)

(14) (A f B) ↔ (¬B f ¬A)

(15) ∀M (M f U)

(16) (U → A) → !A

(17) (U f A) → !A

(18) !!A → !A

(19) $!!A \leftrightarrow !A$

(20) $(U \text{ f } A) \leftrightarrow (A \infty U)$

(21) $!A \leftrightarrow (A \infty U)$

(22) $!V$

(23) $V \infty U$

(23') $V \text{ f } U$

(24) $A \text{ f } A$

(25) $(A \rightarrow B) \rightarrow (A \text{ f } B)$

(26) $(A \leftrightarrow B) \rightarrow (A \infty B)$

(27) $\forall M (\cap \text{ f } \neg M)$

(28) $\cap \text{ f } \cap$

(29) $\cap \text{ f } U$

(30) $\cap \text{ f } \Lambda$

(31) $\cap \infty \Lambda$

(32) $\neg(U \text{ f } \Lambda)$

(33) $\neg(U \rightarrow \Lambda)$

(34) $U \leftrightarrow V$

(35) $\cap \leftrightarrow \Lambda$.

6 'Surprising Consequences' and conceptual objects

Mally sees his theorems (1), (2), (7), (22) and (27)–(35) as 'surprising' (*befremdlich*) or even 'paradoxical'. Of these, (34) and (35) are reported to be the most surprising of his surprising theorems. Some of Mally's explanations concerning the reasons for calling these theorems surprising are also puzzling [12]. Mally's reasons for calling some of his theorems surprising will not be discussed here.

The fact that theorems (23), (34) and (35) are derivable in Mally's system is clearly fatal for it. It has been pointed to by many *deontic* logicians, f.i. Menger, von Wright, Follesdal and Hilpinen, and Lokhorst. Definitely, the system that states, for instance, that every factual state of affairs is obligatory and what is obligatory is the case (34) has no chance to be accepted as a viable system. This totally unacceptable result has a number of confusions as its background and Mally's peculiar idea of what a fact is also belongs to them.

Mally believes that state of affairs, fact and object are distinct from one another. A state of affairs may be taken to be the meaning of a sentence which expresses the corresponding judgment (*Urteil*) but in this case the sentence is different from the proposition taken as propositional function. It is in the former sense that a proposition should be referred to as true (*Tatsachen*)[8] or not true (*Untatsachen*), and this is distinct from how it is referred to in propositional logic, or the theory he calls logistic. '*The concept of factual state of affairs (Tatbestand) lies in the background of the concept of implication, and is a very important concept of thought; it cannot be grasped unless applied to*' [16, p. 289].

Mally insists that logistic is a logic of propositional, or linguistic forms, and is not logical theory proper, for propositional forms themselves never become available for evaluation as being true and false unless some facts, or actual states of affairs, are understood as being described by them. '*This is how the same rules are used for indefinite descriptions of states of affairs and for real facts that indeed may differ from one another*' [16, p. 236]. In other words, Mally explicitly points that to say that a proposition is true is not the same as to say that it describes what is the case, for whereas being true as well as being not true are normally considered to be properties of a proposition, describing what is the case is a property

[8]Mally significantly avoids speaking of truth (Warheit) and truthfulness in his book. The reason for this may be that he has in mind a kind of intensional semantic presuppositions for his system. This may serve an appropriate explanation for the fact that in more or less the same significant way Mally obviates anything analogous to truth-functional logical semantics traditionally used in propositional logic. In doing so he seems to understand such semantic presuppositions as extensional and thus divergent from the semantic presuppositions his object theory requires.

of thought. The idea that our understanding of facts always occurs after the facts themselves occur makes the regularities we notice in material world obsolete, warns Mally in the Foreword of his book. However, notwithstanding that it has occurred as a material truth, once we make a conclusion that something is the case we take the latter as formally necessary (*richtig*). It is in this sense that Mally sees judgment and willing as formally necessary whenever the two meet in fact [16, p. 229].

This is how Mally arrives at his three-fold distinction. There are two groups of entities: propositions which in Mally's version are (contingent) linguistic forms, and states of affairs (*Sachverhalte*) which are kinds of conceptual objects capable of having factual counterparts and which are referred to by sentential variables in the non-deontic part of his system. The states of affairs fall into true and not true.

Throughout chapters I–II Mally diligently avoids telling what true or non-true states of affairs are. His key idea is that logic is a theory about intensionally understood states of affairs (*Sachverhalte*) and thoughts (*Denken*), and it is lies in the very concept of correct thought (*richtige Denken*) that it should grasp and model states of affairs and do so in order to unveil the nature of relations between the states of affairs and conceptual objects (*Gegenstande*) [16, p. 231]. Because applying a thought to a state of affairs in different empirical cases may yield diverse results [16, p. 233], logic should start with investigating what correct thought is. True (*Tatsashe*) states of affairs are those which are referred to by correct thought. It is the relation between the two sorts of conceptual objects that Mally places in the center of his peculiar 'truth theory', namely between what is thought to be a state of affairs and a conceptual object that is meant to correspond to it.[9]

[9] There should be a way of distinguishing between the two in order to establish a sort of correspondence between these kinds of conceptual objects which are clearly distinct for Mally. The borderline has to be looked for in Mally's object theory [22]. Conceptual objects do not necessarily instantiate the properties they consist of; the former may be vague and logically inconsistent with respect to the latter. The fact that such an intensional object matches its factual instantiation is derivative from the fact that the object is sound in logical sense [21].

7 Formalisms for conceptual objects

At this point one may easily notice the need for adequate symbolization for this three-fold division in the non-deontic part of Mally's system, and elementary first-order predicate logic seems to be an appropriate candidate for this. In fact, Mally does an explicit step towards using (monadic) predicate logic when proposing sentential letters as symbols for states of affairs (*Tatbestand*), sentential variables for any state of affairs whatsoever (*Sachverhalt*) and sentential constants V and for true (*Tatsache*) and Λ for not true (*Untatsache*) states of affairs. According to him, the relation between factual (*Falle* x) and conceptual states of affairs ($B(x)$) (*Sachverhalte*) is expressed in the judgment (*Urteil*) of the form

There is at least one x, for which *B(x)* is the case,

which may be evaluated as true or not true and which can be reformulated so as to include all or some x respectively [16, pp. 236–237]. Throughout chapter I he indeed tries to use sentential variables as quantifiers but never goes beyond this point. Unfortunately, already in these symbolisms his three-fold distinction turns *de facto* to be expressed with the help of sentential variables only and thus gets formally collapsed.

However, this is not the only confusion in the non-deontic part of Mally's system. Having introduced the idea of different relations that hold between facts and between conceptual objects Mally apparently should have suggested some symbolisms that would exhibit the difference. His attempts towards doing so are seen in the fact that he introduces material implication which is meant to hold between states of affairs, A $\rightarrow$ B, and a kind of formal implication, A f B, which may be interpreted as the deontic version of standard (aletic) formal implication, according to Def. f. Recalling now that there is a confusion among kinds of objects expressed with the help of sentential variables, letters and constants one gets a confusion with connectives as derived out of the sentential confusion. There seems to be a dilemma: drop the three-fold division and replace it with just one object, either propositional or conceptual, or abandon the functional distinction material\formal for implication. Had

Mally chosen either of the two lines for constructing the non-modal part of his system, this would have prevented it from the collapse just discussed. Disregarding the former and adopting the latter would result in a system of formal implication; following the opposite line, namely holding the former but not the latter leads to a first-order predicate system. Moreover, disregarding both gives a version of classical propositional logic which could have served as a basis for a kind of standard deontic logic. Mally adopts both in his system and this leads him to further confusions.

8 Correct will as conceptual object

Judgments and volitions both refer to the states of affairs but they do so in a different way, says Mally in the Introduction of his book, and he proposes his Deontik as a theory of correct volition [16, pp. 233–234] as distinct from correct judgment which is studied in non-modal logic.[10] Mally believes that human volitions are also conceptual objects and they stem out of definite state of affairs, for what is being wished in them is nothing but some other states of affairs which can be true (*Tatsache*) or not true (*Untatsache*) [16, p. 279]. These conceptual objects serve as the content of intentionally directed acts, for intentional acts are never deprived of their content. Because of the fact that there is always a conceptual object that is determined by some properties, human beings' volitions may be rationally vague and inconsistent in logical sense but this does not exclude a logical possibility for them to contingently realize [16, p. 278].[11]

Contrary to not true states of affairs their true counterparts are capable of having their factual instantiations and this is so due

[10]'Even though Mally regarded many of his theorems as surprising, he thought that he had discovered an interesting concept of 'correct willing' (*richtiges Wollen*) or 'willing in accordance with the facts' which should not be confused with the notions of obligation and willing used in ordinary discourse. Mally's 'exact system of pure ethics' was mainly concerned with this concept, but we will not describe this system because it belongs to the field of ethics rather than deontic logic.'[12]

[11]'The improper will is an obvious demonstrative experience, for the improper ought that wants to be an equivalent to the true state of affairs seems itself to be so only indirectly, namely through the reasoning which points to something together with what out of which true state of affairs follow'.

to the fact that these conceptual objects are always complete and consistent. *'Inasmuch logic does pursue inconsistent propositions neither, so does not the Deontik in what concerns inconsistent and untrue obligations'* [16, p. 248]. Only the volitions that are complete and consistent may realize as true states of affairs. *'Consistency is the key property of correct thought (richtigen Denken) and right will'* [16, p. 244]. This is the reason why Mally holds the view that any correct volition should include all its implicates. In many places of his book Mally insists that his *Deontik* is a logical theory of correct volitions. This is also plain in his Axiom III and Theorems (6), (7), (9). Inferential totality of correct volition is echoed in Mally's idea of human responsibility [16, p. 273. Cf. note 4 above].

There are two other notable features of Mally's conception of willing: that it is agential, but impersonal, and that it is conditional, but in a very special way. Let us consider these properties in turn.

Mally proposes symbolism !A as 'A ought to be the case' (A *soll sein*) or as 'let A be the case' (*es sei* A) in the sense that A is a state of affairs which is being wished by someone [16, p. 241]. Many Mally's critics point to the fact that traditional deontic O-symbol — be it taken as a connective, or as a sentential operator — is seldom read in this way [12; 10, pp. 5–6]. Mally's !A is agential, but impersonal and is goal oriented, but not action-dependent. Therefore, it is small wonder that deontic O-symbol is seldom read in the way Mally introduces his !A, for the two are originally meant to symbolize different entities.

Mally's !A explicitly points to an obligatory state of affairs and seems to reject being interpreted as an obligatory action. On the other hand, Mally uses his !A to express someone's volition which because of being desired becomes someone's ought rather than obligation. Mally's philosophical insights into the nature of volition (*Wollen*) show that it is because of the fact that a volition is always human volition it may become human ought in the sense that it performs as a goal for human conduct at issue [16, pp. 303–306]. It is in this particular agential sense that Mally's willing (*Wollen*) is transformed by an agent into a kind of its personal ought (*Sollen*) when choosing its particular strategy. Therefore, Mally's !A is much closer to the concept of agential choice and to an indeterministic interpretation of agential strategic goal of the kind suggested in *stit*-theories [11], or to what is understood by *tactics* in the deontic logic of A.S. Esenin-Volpin [1], then to deterministic obligations of the sort pursued the framework of the SDL and its developments. In fact, it may be shown that *stit*-reconstruction of Mally's *Deontik* is also possible with the help of some minor renovations to his system. This is why interpreting Mally's !A as standard O-obligation would hardly be the best choice.

From what has been said above concerning his concept of willing it is clear now that only correct volition may turn into agential ought. Def. f and Axiom III suggest that what is taken to be a formal condition for the volition is also should be regarded as a part of the intensional content of the volition at issue. In other words, logical antecedent of the desired state of affairs expressing the (pre)condition of the volition is also part of this volition. On the other hand, Theorem 5 says that correct volition is implied by any state of affairs whatsoever [16, p. 261]. Consequently, due to the ideas that correct volition is implied by any state of affairs and that it should include all the consequences of the desired state of affairs, the concept of correct volition results in unconditional and impersonal volition notwithstanding its start as conditional and human depending. In the beginning Mally uses distinct symbolisms to express ought and obligation: oughts as they are introduced by Mally in the first three postulates are different from obligations given in the Axioms V and IV.

In doing so Mally again starts with proposing important distinctions, namely, between agential strategic ought and conditional obligation, but because of his idea of correct volition which should not only imply all its consequences but its antecedents as well, he finally drops the distinction just introduced and arrives at an ill-formed mixture of the two. This confusion results in a problematic outcome that in his philosophical explanations Mally's volition (*Wollen*) is gradually transformed into ought (*Sollen*) [16, p. 276 and ff],[12] what apparently does conform with what he says when introducing

[12]'In terms of the will, it lies in the very sense of the volition, that to say that the desired state of affairs ought to be is to say that there ought to be any state of affairs in absence of which that which is desired may not happen'.

his !A [16, p. 241][13] in agential indeterministic perspective. But after that, when the idea that agential indeterministic ought is said to include all its consequents, but to follow to the state of affairs which has the predominant chances to happen,[14] things *de facto* go wrong and agential ought is turned into a sort of unconditional obligation [16, pp. 299–301, axiom IV].[15] Thus, at this point the distinction is corrupted. Mally is aware of these transformations and he notoriously purports to explain them by pointing to his idea that in order to be capable of being realized the volition should be complete and consistent, but this consideration does not help much here.

Having introduced his illuminative and fertile of further developments concept of agential volition !A, Mally could have subsequently developed a kind of indeterministic logic of norms, had he abandoned the idea that correct volition should imply all the consequents of itself, or keep to what is logically necessary, but he did not. On the basis of the same concept he also could have developed a kind of (non-agential) deterministic deontic logic, had he dropped the idea that correct volition should include all its antecedents, or stem out of the definite state of affairs, but he did not, too. Instead he preserved both and once again arrived at a collapsing confusion of ought and obligation in the deontic part of his *Deontik*. This is particularly the reason for his 'strange' Theorem 22 which yields yet more strange consequences.

[13]'This ought, precisely the ought of the definite state of affairs, corresponds to the will as to a counterpart conceptual object: it describes the object, namely the state of affairs, to which the will is directed'.

[14]'This is how the judgment and the subsequent decision come to be correct: they are materially correct, if they both keep to the true state of affairs; they are formally correct, if they have been taken in the sense of the predominant possibility, therefore, have themselves proved to happen' [16, p. 300].

[15]'The requirement of formal correctness, enjoining what is of man as a volitional essence requires and may be reasonably required: to satisfy the requirements of substantive correctness to the best of knowledge. The requirements of formally correct will specify an ideal: to fulfill the aim, which cannot be required as necessary proper, it is necessary to keep to these requirements rigorously and unconditionally' [16, p. 301].

9 Unconditional obligation

Apart from volitions Mally introduces another kind of obligation, unconditional obligation U (*das unbedingt Geforderte, das Sollensgemaesse*) which is seen as distinct from !A because of the fact that the latter is agential and conditional. Unconditional constant U and its negation ∩ seem to be very close to Kangerian Q-constant which is meant to express a normative code. The difference between Kangerian Q and Mally‘s U lies in the idea that Mally takes his deontic constant to refer to the obligatory states of affairs whereas Kangerian Q depict an actual normative code.

It is tempting to call the negation of U, ∩, unconditionally forbidden (*das unbedingt Verbotene*). In some places of his book Mally occasionally does so [16, p. 296–297], but as a whole the notions of forbidden and permitted are not to be found in his book. The reason for this according to Mally lies in two important facts concerning the issue. The first is that unconditionally obligatory is a conceptual object and as such is necessarily applied to a state of affairs. The second is that unconditional obligation is agent-free, or agent-independent. The concepts of permission and prohibition seem not to belong to the domain of the *Deontik* which is seen as logic of correct willing. In this system, prohibition (*verkert*-U, *Sollenswidrige*) is just the counterpart of unconditionally obligatory and is not an object [16, p. 250].

When introducing his deontic constant U Mally speaks of a kind of positive obligation and its negation and he takes the former to be consistent and actual unconditionally obligatory state of affairs, though, perhaps, in his view of conceptual objects to say that a state of affairs is consistent and actual would be redundant. Unconditionally obligatory never implies what is incompatible to it, or its negation. Consequently, the negation of U, or ∩, is principally unobtainable as a state of affairs because of its inconsistency. This leads us to the conclusion that it is not quite correct to take Mally's *deontic* constant ∩ as unconditionally forbidden of the same sort as unconditionally obligatory.

Agent-dependency is the background of the distinction between Mally's ought and (unconditional) obligation. He draws a borderline between obligation which he regards as definitive and independent of personal representations (*objective Bestimmtheit*) in the sense of (2) and ought which is an agential goal and thus may be vague and even inconsistent (*subjective Unbestimmtheit*) [16, pp. 280–281] in the sense of (3)–(4). Axiom IV says that social agents in outlining the trends of their behavior take into account that there exist some (legal or moral) norms. This is not to say that these obligations exist independently of agents, but just that they are not agential ones.

In his purports to introduce a kind of deontologically understood obligation, Mally apparently falls into the trap which he has prepared for himself when diligently avoiding any semantic considerations and carelessly following his object theory. In the deontic part of his system (Axiom IV) Mally would like to propose an objectively interpreted obligation and for this reason he speaks of !U — 'unconditional demand as a principle of actuality of obligation (*Grundsatz der Tatsaechlichkeit des Sollens*)' ∃U !U [16, p. 249] — not only as of existing independently of social agents, but just of the one that exist.

And the trap shuts. Indeed, his idea of true states of affairs as logically consistent conceptual objects taken together with the corrupted formal distinction between the conceptual objects which are capable of having factual instantiations and those that are not (see section 7 and note 9 above) results in that formally there is simply no room for any other kinds of conceptual objects to be taken as existing in the non-modal part of his *Deontik*.[16]

Neither there is any in the deontic part of his system. Agential ought, when taken altogether with all its consequents in the deterministic way (see section 8 above), clearly overlaps with unconditional obligation. This is the philosophical background for Mally's 'most surprising consequences' (34)–(35), which follow out of his Theorem 22 [16, p. 269].

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[16] This is particularly why Lokhorst sees Mally's Axiom IV as redundant [12]. However, his suggestion to replace it with !U will turn Mally's unconditional obligation into unconditional agential ought.

10 Conclusion

In his *Deontik* Mally has made a number of powerful distinctions significant in what regards logic of norms. These are the distinctions between

a) Actual, or material, and intensional, or formal states of affairs;

b) Material and formal implications as holding in the case of two mentioned in a);

c) Agential ought and obligation.

All the distinctions are grounded in his object theory and have proved to be crucial to the development of the logic of norms after Mally. In 1926, the ideas of (a) and (b) were already known in the logical community [6], but he is apparently a pioneer of the idea of (c)-distinction which has started to be developed in the logical systems of norms in the last decades of the XXth century. Unfortunately, in his formalisms he got all the three distinctions confused and this led to the system's collapse. However, in the paper, more evidence is suggested to support the thesis that *Mally's pioneering effort deserves rehabilitation rather than contempt* [12].

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What trends in non-classical logic were anticipated by Nikolai Vasiliev?

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abstract. In this paper we discuss a question about the trends in non-classical logic that were exactly anticipated by Nikolai Vasiliev. We show the influence of Vasiliev's Imaginary logic on paraconsistent logic. Metatheoretical relations between Vasiliev's logical systems and many-valued predicate logics are established. We also make clear that Vasiliev has developed a sketch of original system of intensional logic and expressed certain ideas of modal and temporal logics.

Keywords: Nikolai Vasiliev, imaginary logic, syllogistic, paraconsistent logic, many-valued logic, intensional logic, modal logic, temporal logic

A century has passed since the publication of logical works of outstanding Russian logician Nikolai Vasiliev. Now he is rightfully considered to be the founder of non-classical logic together with Łukasiewicz, Lewis, and Brouwer. Vasiliev has published only three papers on logic between 1910 and 1913, but these papers gave long lasting effect on the development of non-classical logic in 20-th century.

There are various opinions on the subject of what trends in Non-classical logic were exactly anticipated by Nikolai Vasiliev.

Vasiliev is ordinary regarded to be the forerunner of paraconsistent logic. It is true that Vasiliev's idea to construct a consistent logic of contradictory world (Imaginary, non-Aristotelian logic) is really in keeping with some informal attitudes of paraconsistent logic.

Before paraconsistent logic was set up, some researchers considered Vasiliev's works to contain the ideas of many-valued logic. The reason for this opinion was that Vasiliev introduced the third form of propositions ('indifferent', or contradictory statements) along with affirmative and negative statements. Besides, Vasiliev put forward an idea of logic of n dimensions which has n initial qualities of propositions.

We'll also make clear that Vasiliev has developed a sketch of original system of intensional logic and expressed certain ideas of modal and temporal logics.

1 Imaginary logic: consistent logic of contradictory world

The best known Vasiliev's logical system is his Imaginary non-Aristotelian logic — one of the first ever non-classical logical theories.

Inspired by the ideas of non-Euclidean geometry contributed by his colleague from Kazan University Nilolai Lobachevski, Vasiliev in his paper 'Imaginary (non-Aristotelian) logic' [1, pp. 53–93] constructed a deductive theory of syllogistic kind. The language of this theory contains besides affirmative and negative propositions contradictory (so called indifferent) ones with syllogistic copula 'is and is not simultaneously'. According to Vasiliev, such propositions are false in our terrestrial world but can turn to be true in a certain imaginary world.

A. Arruda was the first who appreciated Vasiliev as a forerunner of paraconsistent logic. On the basis of Vasiliev's ideas she formulated three propositional calculi **V1**–**V3** useful as a logical part of non-trivial inconsistent theories [7].

V3 calculus is the most close to Vasiliev's Imaginary logic. It includes standard negation ($\neg$) and conjunction (&) together with their non-classical analogs $^{-}$ and $\cdot$. In contrast to classical connectives, non-classical negation ($^{-}$) can be applied only to propositional variables, and non-classical conjunction ($\cdot$) concatenates a variable and its negation: $\bar{\gamma}$ and $\gamma \cdot \bar{\gamma}$ are the formulas if γ is a propositional variable. In the system **V3** formulas γ, $\bar{\gamma}$, $\gamma \cdot \bar{\gamma}$ are pairwise incompatible, and the law of excluded forth $\gamma \vee \bar{\gamma} \vee \gamma \cdot \bar{\gamma}$ is valid.

However, **V3** can not be regarded as an adequate formalization of Vasiliev's Imaginary logic. The language of propositional logic is too poor to solve this problem. Imaginary logic was formulated by Vasiliev as a syllogistic of special kind, and the law of excluded forth is valid here only for singular propositions.

Vasiliev himself singled out the following types of basic propositions in Imaginary logic (v is an arbitrary singular term, S and P are any universal terms):

(1) *singular*:
'v is P' (we'll use for them symbolic notation $\mathbf{J_1}vP$),
'v is not P' – $\mathbf{J_2}vP$,
'v is and is not P' – $\mathbf{J_3}vP$;

(2) *universal*:
'Every S is P' – $\mathbf{A_1}SP$,
'Every S is not P' – $\mathbf{A_2}SP$,
'Every S is and is not P' – $\mathbf{A_3}SP$;

(3) *definite particular*:
'Some S are P, and all the rest of S are not P' – $\mathbf{T_1}SP$,
'Some S are P, and all the rest of S are and are not P' – $\mathbf{T_2}SP$,
'Some S are not P, and all the rest of S are and are not P' – $\mathbf{T_3}SP$,
'Some S are P, some S are not P, and all the rest of S are and are not P' – $\mathbf{T_4}SP$.

In addition Vasiliev used *indefinite particular* propositions:
'Some S are P' – $\mathbf{I_1}SP$,
'Some S are not P' – $\mathbf{I_2}SP$,
'Some S are and are not P' – $\mathbf{I_3}SP$.

T. Kostyuk and V. Markin [3] constructed the calculus **IL** with initial constants $\mathbf{J_1}$, $\mathbf{J_2}$, $\mathbf{J_3}$, $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3}$ that is an adequate formalization of Vasiliev's Imaginary logic. **IL** contains the following deductive postulates.

Axiom schemes:

A0. Propositional tautologies,
A1. $\neg(\mathbf{J_1}vP \mathbin{\&} \mathbf{J_2}vP)$,
A2. $\neg(\mathbf{J_1}vP \mathbin{\&} \mathbf{J_3}vP)$,
A3. $\neg(\mathbf{J_2}vP \mathbin{\&} \mathbf{J_3}vP)$,
A4. $\mathbf{J_1}vP \vee \mathbf{J_2}vP \vee \mathbf{J_3}vP$,
A5. $(\mathbf{J_1}vP \mathbin{\&} \mathbf{J_1}vS) \supset \mathbf{I_1}SP$,
A6. $(\mathbf{J_2}vP \mathbin{\&} \mathbf{J_1}vS) \supset \mathbf{I_2}SP$,
A7. $(\mathbf{J_3}vP \mathbin{\&} \mathbf{J_1}vS) \supset \mathbf{I_3}SP$,
A8. $\mathbf{I_1}SS$.

Rules:

R1. $\dfrac{A \supset B, A}{B}$, **R3**. $\dfrac{(\mathbf{J_1}vS\ \&\ \mathbf{J_2}vP) \supset A}{\mathbf{I_2}SP \supset A}$,

R2. $\dfrac{(\mathbf{J_1}vS\ \&\ \mathbf{J_1}vP) \supset A}{\mathbf{I_1}SP \supset A}$, **R4**. $\dfrac{(\mathbf{J_1}vS\ \&\ \mathbf{J_3}vP) \supset A}{\mathbf{I_3}SP \supset A}$

(in **R2**–**R4** the term v does not occur in A).

Definitions of universal and definite particular propositions:
$\mathbf{A_1}SP \rightleftharpoons \neg\mathbf{I_2}SP\ \&\ \neg\mathbf{I_3}SP$,
$\mathbf{A_2}SP \rightleftharpoons \neg\mathbf{I_1}SP\ \&\ \neg\mathbf{I_3}SP$,
$\mathbf{A_3}SP \rightleftharpoons \neg\mathbf{I_1}SP\ \&\ \neg\mathbf{I_2}SP$,
$\mathbf{T_1}SP \rightleftharpoons \mathbf{I_1}SP\ \&\ \mathbf{I_2}SP\ \&\ \neg\mathbf{I_3}SP$,
$\mathbf{T_2}SP \rightleftharpoons \mathbf{I_1}SP\ \&\ \neg\mathbf{I_2}SP\ \&\ \mathbf{I_3}SP$,
$\mathbf{T_3}SP \rightleftharpoons \neg\mathbf{I_1}SP\ \&\ \mathbf{I_2}SP\ \&\ \mathbf{I_3}SP$,
$\mathbf{T_4}SP \rightleftharpoons \mathbf{I_1}SP\ \&\ \mathbf{I_2}SP\ \&\ \mathbf{I_3}SP$.

Formal counterparts of all the laws of Imaginary logic which Vasiliev marked out are provable in **IL**.

The semantics of **IL** proposed by T. Kostyuk and V. Markin [3] is based on assignment several extensional characteristics to each universal term — its extension, anti-extension and contradictory domain. Such an idea was implicitly presented in Vasiliev's text.

Define **IL**-model as follows: $< \mathbf{D}, \varphi, \psi_1, \psi_2, \psi_3 >$, where $\mathbf{D} \neq \emptyset$, $\varphi(v) \in \mathbf{D}, \psi_1, \psi_2, \psi_3$ are functions which put in correspondence every universal term P with a subset of $\mathbf{D}$ and satisfy the following conditions: $\psi_1(P) \neq \emptyset, \psi_1(P) \cap \psi_2(P) = \emptyset, \psi_1(P) \cap \psi_3(P) = \emptyset, \psi_2(P) \cap \psi_3(P) = \emptyset, \psi_1(P) \cup \psi_2(P) \cup \psi_3(P) = \mathbf{D}$.

Informally, $\psi_1(P)$ is an extension of P, $\psi_2(P)$ is an anti-extension of P, and $\psi_3(P)$ is a contradictory domain with respect to P.

Truth definitions for atomic formulas in a model $< \mathbf{D}, \varphi, \psi_1, \psi_2, \psi_3 >$:
$| \mathbf{J_1}vP |= \mathbf{1}\ \ iff\ \ \varphi(v) \in \psi_1(P)$,
$| \mathbf{J_2}vP |= \mathbf{1}\ \ iff\ \ \varphi(v) \in \psi_2(P)$,
$| \mathbf{J_3}vP |= \mathbf{1}\ \ iff\ \ \varphi(v) \in \psi_3(P)$,
$| \mathbf{I_1}SP |= \mathbf{1}\ \ iff\ \ \psi_1(S) \cap \psi_1(P) \neq \emptyset$,
$| \mathbf{I_2}SP |= \mathbf{1}\ \ iff\ \ \psi_1(S) \cap\ \psi_2(P) \neq \emptyset$,
$| \mathbf{I_3}SP |= \mathbf{1}\ \ iff\ \ \psi_1(S) \cap\ \psi_3(P) \neq \emptyset$.

Truth definitions for complex formulas are usual.

It can easily be shown that the truth conditions for the forms of universal propositions are the following:

$\mid \mathbf{A_1}SP \mid = \mathbf{1}\ \ iff\ \ \psi_1(S) \subseteq \psi_1(P)$,
$\mid \mathbf{A_2}SP \mid = \mathbf{1}\ \ iff\ \ \psi_1(S) \subseteq \psi_2(P)$,
$\mid \mathbf{A_3}SP \mid = \mathbf{1}\ \ iff\ \ \psi_1(S) \subseteq \psi_3(P)$.

A formula A is *true in a model* $< \mathbf{D}, \varphi, \psi_1, \psi_2, \psi_3 >\ \ iff\ \ \mid A \mid = \mathbf{1}$ in this model. A formula A is *valid* $\ iff\ $ it is true in every model.

The adequacy of the semantics for **IL** was proved by T. Kostyuk in her Ph.D. thesis 'Reconstruction of N.A. Vasiliev's logical systems by means of modern logic' defended in Lomonosov Moscow State University in 1999.

2 Logic of n dimensions and n-valued logic

Some researchers (L. Chwistek, A.N. Maltsev, G.N. Kline, N. Rescher, M. Jammer, V.V. Anosova) considered Vasiliev to be a predecessor of many-valued logic. It appears that such an opinion is grounded on the three types of propositions' quality in his Imaginary logic. Moreover, in the paper 'Imaginary (non-Aristotelian) logic' Vasiliev advanced an idea of possible development of the logic of n dimensions [1, pp. 76–77]. For him, such systems differ in a number of types of propositions varying in quality. Aristotelian syllogistic is bidimensional, imaginary logic has three dimensions. In general, a logic of n dimensions must contain n types of propositions with different qualities. Vasiliev himself did not develop these idea into a logical theory.

The reconstruction of the logic on n dimensions was realized by T. Kostyuk [2]. She formulated an exact and intuitively transparent semantics for syllogistic language with n types of propositions varying in quality along with the adequate axiomatization.

The system **IL** can be in a natural way extended to syllogistics $\mathbf{IL_n}$ with arbitrary number of propositions with different qualities.

There are n syllogistic constants for singular $(\mathbf{J_1}, \mathbf{J_2}, \ldots, \mathbf{J_n})$, universal $(\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n})$ and indefinite particular $(\mathbf{I_1}, \mathbf{I_2}, \ldots, \mathbf{I_n})$ propositions of different quality. Let $\mathbf{J_i}vP$ means that an individual v stands in i-th qualitative relation to P, $\mathbf{A_1}SP$ — every object from S stands in i-th qualitative relation to P, $\mathbf{I_1}SP$ — some object from S stands in i-th qualitative relation to P. When $i = 1$ we have a form of affirmative proposition with corresponding quantity. It is convenient to suppose the formulas with $i = n$ to be the forms of negative propositions.

$\mathbf{IL_n}$-*model* is a structure $< \mathbf{D}, \varphi, \psi_1, \psi_2, \ldots, \psi_n >$, where $\mathbf{D} \neq \emptyset$, $\varphi(v) \in \mathbf{D}, \psi_i(P) \subseteq \mathbf{D}, \psi_1(P) \neq \emptyset, \psi_i(P) \cap \psi_j(P) = \emptyset$, where $1 \leq i, j \leq n$ and $i \neq j$; $\psi_1(P) \cup \psi_2(P) \cup \ldots \cup \psi_n(P) = \mathbf{D}$. In this semantical framework each universal term is connected with n extensional characteristics.

The truth definitions for atomic formulas are the following:

$| \mathbf{J_i} vP |= \mathbf{1} \ \ iff \ \ \varphi(v) \in \psi_i(P)$,
$| \mathbf{A_i} SP |= \mathbf{1} \ \ iff \ \ \psi_1(S) \subseteq \psi_i(P)$,
$| \mathbf{I_i} SP |= \mathbf{1} \ \ iff \ \ \psi_1(S) \cap \psi_i(P) \neq \emptyset$.

Truth definitions for complex formulas are usual.

A formula A is *true in a model* $< \mathbf{D}, \varphi, \psi_1, \psi_2, \ldots, \psi_n >$ iff $| A |= \mathbf{1}$ in this model. A formula A is *valid* iff it is true in every model.

The set of $\mathbf{IL_n}$-valid formulas is axiomatized by the calculus $\mathbf{IL_n}$ with initial syllogistic constants $\mathbf{J_1}, \mathbf{J_2}, \ldots, \mathbf{J_n}, \mathbf{I_1}, \mathbf{I_2}, \ldots, \mathbf{I_n}$. Universal propositions can be defined as follows:

$\mathbf{A_i} SP \rightleftharpoons \ \&_{\mathbf{j \neq i}} \neg \mathbf{I_j} SP$.

There are the following deductive postulates in $\mathbf{IL_n}$:

A0. Propositional tautologies,
A1. $\neg(\mathbf{J_i} vP \ \& \ \mathbf{J_j} vP)$, where $i \neq j$,
A2. $\mathbf{J_1} vP \vee \mathbf{J_2} vP \vee \ldots \vee \mathbf{J_n} vP$,
A3. $(\mathbf{J_i} vP \ \& \ \mathbf{J_1} vS) \supset \mathbf{I_i} SP$,
A4. $\mathbf{I_1} SS$,
R1. $\dfrac{A \supset B, A}{B}$,
R2. $\dfrac{(\mathbf{J_1} vS \ \& \ \mathbf{J_i} vP) \supset A}{\mathbf{I_i} SP \supset A}$ (v does not occur in A).

The semantical adequacy for $\mathbf{IL_n}$ was proved by T. Kostyuk [2].

System **IL** turned to be three-dimensional case of many-dimensional logic, while a two-dimensional case is presented by the system of traditional syllogistic with singular terms [6] that is the extension of well-known Łukasiewicz' syllogistic.

It should be mentioned that the appearance of a proposition of a new quality does not support by itself the revision of principle of two-valuedness. Vasiliev did not consider the possibility of the third value seriously. He preferred to operate with classical valuations 'true' and 'false'.

In what follows we consider the issue of connection between Vasiliev's logical legacy and many-valuedness in a different manner — as a problem of metalogical relationship between logic of n dimensions and many-valued logic.

In [5] we proposed an intuitively clear and simple adequate translation of Imaginary logic (**IL** calculus) into the quantified three-valued logic and proved that this translation is an embedding.

This result was generalized to an arbitrary logic of n dimensions by Igor Alexeev in his graduation thesis 'Vasiliev's logic of n dimensions and many-valued predicate logic', prepared at the Department of Logic, Lomonosov Moscow State University in 2009.

He has showed that axiomatic calculus $\mathbf{IL_n}$, formalizing logic of n dimensions, is embedded into monadic n-valued predicate logic with the following properties:

(1) **j**-operators are expressible for any possible value;

(2) standard propositional connectives take the same values for classical arguments ($\mathbf{1}$, $\mathbf{0}$) as in classical logic;

(3) formulas of the type $\forall\alpha A$ take the value $\mathbf{1}$ iff for arbitrary value of α, the value of A is $\mathbf{1}$, and take the value $\mathbf{0}$ iff for some value of α, A takes the value $\mathbf{0}$;

(4) formulas of the type $\exists\alpha A$ take the value $\mathbf{1}$ iff for some value of α, A takes the value $\mathbf{1}$, and take the value $\mathbf{0}$ iff for arbitrary value of α, the value of A is $\mathbf{0}$.

An obvious example of such a system is quantified n-valued Łukasiewicz' logic $\mathbf{Ł_n}$.

$\mathbf{Ł_n}$-model is a structure $< \mathbf{D}, \varphi, \psi_1, \psi_2, \ldots, \psi_n >$, where $\mathbf{D} \neq \emptyset$, $\varphi(v) \in \mathbf{D}, \psi_1, \psi_2, \ldots, \psi_n$ are the functions which put in correspondence every predicate symbol P with a subset of $\mathbf{D}$ and satisfy the following conditions: $\psi_i(P) \cap \psi_j(P) = \emptyset$, for any $i \neq j$ from 1 up to n; $\psi_1(P) \cup \psi_2(P) \cup \ldots \cup \psi_n(P) = \mathbf{D}$.

Let $\mathbf{g}$ be an assignment for variables: $\mathbf{g}(\alpha) \in \mathbf{D}$ for arbitrary variable α.

The set of possible values for formulas is $\{\mathbf{1}, \frac{\mathbf{n-2}}{\mathbf{n-1}}, \ldots, \frac{\mathbf{1}}{\mathbf{n-1}}, \mathbf{0}\}$.

Valuation for terms and formulas is defined as follows:

$\mathbf{V_g}(\alpha) = \mathbf{g}(\alpha), \qquad \mathbf{V_g}(v) = \varphi(v),$

$\mathbf{V_g}(Pt) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$ iff $\mathbf{V_g}(t) \in \psi_i(P),$

$\mathbf{V_g}(\neg A) = \mathbf{1} - \mathbf{V_g}(A),$
$\mathbf{V_g}(A \& B) = \min(\mathbf{V_g}(A), \mathbf{V_g}(B)),$
$\mathbf{V_g}(A \vee B) = \max(\mathbf{V_g}(A), \mathbf{V_g}(B)),$
$\mathbf{V_g}(A \supset B) = \min(\mathbf{1}, \mathbf{1} - \mathbf{V_g}(A) + \mathbf{V_g}(B)),$

$\mathbf{V_g}(\forall \alpha A) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$ iff $\mathbf{V_{g'}}(A) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$ for some $\mathbf{g'} =_\alpha \mathbf{g}$, and there is no $\mathbf{g'} =_\alpha \mathbf{g}$ such that $\mathbf{V_{g'}}(A) < \frac{\mathbf{n-i}}{\mathbf{n-1}}$,

$\mathbf{V_g}(\exists \alpha A) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$ iff $\mathbf{V_{g'}}(A) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$ for some $\mathbf{g'} =_\alpha \mathbf{g}$, and there is no $\mathbf{g'} =_\alpha \mathbf{g}$ such that $\mathbf{V_{g'}}(A) > \frac{\mathbf{n-i}}{\mathbf{n-1}}$

($\mathbf{g'} =_\alpha \mathbf{g}$ means the following: $\mathbf{g'}$ differs from $\mathbf{g}$ at most in assignment for α).

$\mathbf{j}$-operators are expressible in $\mathbf{Ł_n}$ by the following interpretation:

$\mathbf{V_g}(\mathbf{j_i}A) = \mathbf{1}$, if $\mathbf{V_g}(A) = \frac{\mathbf{n-i}}{\mathbf{n-1}}$, otherwise – $\mathbf{V_g}(\mathbf{j_i}A) = \mathbf{0}$.

Formula A is *valid in* $\mathbf{Ł_n}$*-model* iff $\mathbf{V_g}(A) = \mathbf{1}$, for any assignment $\mathbf{g}$. Formula A is *valid* iff A is valid in any $\mathbf{Ł_n}$-model.

The embedding operation from logic of n dimensions $\mathbf{IL_n}$ into quantified n-valued logic $\mathbf{Ł_n}$ is defined in two stages.

First define the mapping * of the set of $\mathbf{IL_n}$-formulas into the set of formulas of quantified n-valued Łukasiewicz' logic:

$(\mathbf{J_i}vP)^\star = \mathbf{j_i}Pv,$
$(\mathbf{I_i}SP)^\star = \exists x(\mathbf{j_1}Sx \ \& \ \mathbf{j_i}Px),$
$(\neg A)^\star = \neg A^\star,$
$(A \otimes B)^\star = A^\star \otimes B^\star$, where $\otimes$ is any binary connective.

Further on the basis of * define the embedding operation Θ:

$$\Theta(A) = (\exists x \mathbf{j_1} S_1 x \ \& \ \exists x \mathbf{j_1} S_2 x \ \& \ \ldots \ \& \ \exists x \mathbf{j_1} S_m x) \supset A^\star,$$

where A is an arbitrary formula of $\mathbf{IL_n}$ language, and $S_1, S_2, \ldots, S_m$ is the list of all universal terms in A.

This result shows the existence of a natural interpretation of any Vasiliev's n-dimensional logic (including his Imaginary logic) in a quantified many-valued logic's framework.

3 Imaginary logic as intensional logic

In the final part of the paper 'Imaginary (non-Aristotelian) logic' Vasiliev made an attempt to formulate intensional semantics for the propositions of his logical system 'Every S is P', 'Every S is not P', 'Every S is and is not P'.

Vasiliev compares Imaginary logic with non-Euclidian geometry and poses a question about possible interpretation of Imaginary logic in terms of our terrestrial world:

> 'We can propose a real interpretation of Non-Euclidian geometry, we can find in our Euclidian space the essences with non-Euclidean geometry... A real interpretation of Lobachevsky's geometry is a geometry of surface with constant negative curvature, of so called pseudo-sphere... In exactly the same way it is possible to find in our world the essences with the logic analogous to imaginary logic' [1, p. 81].

Vasiliev proposed three 'terrestrial' interpretations of Imaginary logic. The core idea of the most interesting interpretation is to associate with each term of a categorical statement not a set of individuals but a *concept* considered as a set of characters and to treat syllogistic constants as denoting intensional relations between concepts. According to this approach, 'Every S is P' means that S contains all characters from P. 'Every S is not P' means that, for an arbitrary character from P, the concept S contains contradictory one, 'Every S is and is not P' means that S contains some characters from P as well as characters which contradict to some others.

Vasiliev emphasized that the logic of concepts based on such semantics differs from the main version of Imaginary logic as well as from the standard syllogistic. For example, some first figure syllogisms with minor negative premise are valid: 'Every M is P. Every S is not M. Hence, every S is not P'.

Vasiliev's ideas, related to this fragment of his paper, were explicated semantically by V. Markin and D. Zaitsev [8].

Let $\mathbf{L}$ be a set of literals — positive and negative characters — $\{p_1, \sim p_1, p_2, \sim p_2, \ldots\}$.

Then a *concept* is an arbitrary non-empty and consistent subset of $\mathbf{L}$, i.e. a set $\alpha \subseteq \mathbf{L}$, which satisfies the following conditions:

$$\text{(i) } \alpha \neq \emptyset; \quad \text{(ii) there is no } p_i\text{: } p_i \in \alpha \text{ and } \sim p_i \in \alpha.$$

Let $\mathbf{M}$ be the set of all concepts. We define an operation * on $\mathbf{M}$, which assigns to every concept α a *contrary* concept α^*:

$$p_i \in \alpha^* \ \ iff \ \ \sim p_i \in \alpha \text{ and } \sim p_i \in \alpha^* \ \ iff \ \ p_i \in \alpha.$$

Vasiliev himself used the same operation:

> 'If the concept A consists of characters p, q, r, s, ... then the concept non-A must consist of characters non-p, non-q, non-r, non-s, and so on' [1, p. 88].

Vasiliev proposed semantical definitions only for universal statements. As before, let $\mathbf{A_1}SP$ be the form of universal affirmative statements 'Every S is P', $\mathbf{A_2}SP$ — the form of universal negative statements 'Every S is not P', and $\mathbf{A_3}SP$ — the form of universal indifferent statements 'Every S is and is not P'.

Let $\mathbf{d}$ be a function assigning arbitrary concepts to terms: $\mathbf{d}(P) \in \mathbf{M}$. Define a valuation associated with $\mathbf{d}$:

$\mid \mathbf{A_1}SP \mid^{\mathbf{d}} = \mathbf{1} \ \ iff \ \ \mathbf{d}(P) \subseteq \mathbf{d}(S)$,
$\mid \mathbf{A_2}SP \mid^{\mathbf{d}} = \mathbf{1} \ \ iff \ \ \mathbf{d}(P)^* \subseteq \mathbf{d}(S)$,
$\mid \mathbf{A_3}SP \mid^{\mathbf{d}} = \mathbf{1} \ \ iff \ \ \mathbf{d}(P) \cap \mathbf{d}(S) \neq \emptyset$ and $\mathbf{d}(P)^* \cap \mathbf{d}(S) \neq \emptyset$.

However to formulate complete system of Imaginary logic one needs more then just universal statements. In the main version of this logic Vasiliev uses as well particular statements: 'Some S are P' ($\mathbf{I_1}SP$), 'Some S are not P' ($\mathbf{I_2}SP$) and 'Some S are and are not P' ($\mathbf{I_3}SP$). V. Markin and D. Zaitsev [8] offer the following truth definitions for the particular propositions:

$\mid \mathbf{I_1}SP \mid^{\mathbf{d}} = \mathbf{1} \ \ iff \ \ \mathbf{d}(P)^* \cap \mathbf{d}(S) = \emptyset$,

$| \mathbf{I_2}SP |^{\mathbf{d}} = \mathbf{1}$ iff $\mathbf{d}(P) \cap \mathbf{d}(S) = \emptyset$,
$| \mathbf{I_3}SP |^{\mathbf{d}} = \mathbf{1}$ iff $\mathbf{d}(P) \setminus \mathbf{d}(S) \neq \emptyset$ and $\mathbf{d}(P)^* \setminus \mathbf{d}(S) \neq \emptyset$,
and usual truth definitions for complex formulas.

A formula is valid in this 'intensional' semantics iff it takes value '**1**' under any assignment **d**.

The set of valid formulas is axiomatized by the calculus **IL2** containing propositional tautologies and axiom schemes:

A1. $(\mathbf{A_1}MP \& \mathbf{A_1}SM) \supset \mathbf{A_1}SP$, **A10**. $\neg(\mathbf{A_1}SP \ \& \ \mathbf{I_2}SP)$,
A2. $(\mathbf{A_1}MP \& \mathbf{A_2}SM) \supset \mathbf{A_2}SP$, **A11**. $\neg(\mathbf{A_2}SP \ \& \ \mathbf{I_1}SP)$,
A3. $(\mathbf{A_2}MP \& \mathbf{A_1}SM) \supset \mathbf{A_2}SP$, **A12**. $\mathbf{I_1}SP \supset \mathbf{I_1}PS$,
A4. $(\mathbf{A_2}MP \& \mathbf{A_2}SM) \supset \mathbf{A_1}SP$, **A13**. $\mathbf{I_2}SP \supset \mathbf{I_2}PS$,
A5. $(\mathbf{A_1}MP \& \mathbf{I_1}SM) \supset \mathbf{I_1}SP$, **A14**. $\mathbf{A_1}SP \supset \mathbf{I_1}SP$,
A6. $(\mathbf{A_1}MP \& \mathbf{I_2}SM) \supset \mathbf{I_2}SP$, **A15**. $\mathbf{A_2}SP \supset \mathbf{I_2}SP$,
A7. $(\mathbf{A_2}MP \& \mathbf{I_1}SM) \supset \mathbf{I_2}SP$, **A16**. $\mathbf{A_3}SP \equiv \neg\mathbf{I_1}SP \& \neg\mathbf{I_2}SP$,
A8. $(\mathbf{A_2}MP \& \mathbf{I_2}SM) \supset \mathbf{I_1}SP$, **A17**. $\mathbf{I_3}SP \equiv \neg\mathbf{A_1}SP \& \neg\mathbf{A_2}SP$.
A9. $\mathbf{A_1}SS$,

The only rule is *modus ponens*.

Thus, Vasiliev has developed a sketch of the alternative version of Imaginary logic based on intensional interpretation of its propositions. He showed the manifold of non-classical logical systems, which are formulated in the same language and differ from each other in sets of laws.

4 Some ideas of modal and temporal logics

In his first paper 'On particular statements, the triangle of oppositions, the law of excluded forth' Vasiliev proposed to treat singular statements as temporal i.e. containing either temporal parameter or temporal characteristic. Vasiliev differentiates two kinds of singular statements: 'statements on the fact' and 'statements on the concept'.

Singular statements on the fact refer to an individual in the certain moment in time, to definite state of the individual in the history of its existence:

> 'The copula of such statements presumes the exact designation of temporal moment, for the subjects of such

> singular statements — perceptions and mental representations — always refer to the certain moment of time' [1, p. 51].

Vasiliev gives the following examples of the statements on the fact: 'Ivan Ivanovich is drunk now', '*NN* passed away at 5 a.m. yesterday', '*NN* is sick today'.

The subject of the statements on the concept represents the set of all possible states of an individual over the time of its existence:

> 'The subject of singular statement Caesar, Goethe etc. can be a concept, and then it symbolizes all terrestrial life of Caesar and Goethe, it subordinates the set of certain moments of Caesar's and Goethe's life to the unity of the concept' [1, p. 51];
>
> 'All these certain moments in Caesar's life: Caesar with the robbers, Caesar as the conqueror of Vercingetorix, Caesar as the monarch, Caesar as the lover of Cleopatra, Caesar killed with conspirators' dagger, — all of them are symbolized in the united concept 'Caesar' in the same way as Caesar, Pompeius and Gaius are symbolized in the united concept of human being' [1, p. 50].

Then there are three kinds of the statements on the concept: (1) an individual always has a property, (2) an individual never has a property, (3) an individual sometimes has a property, and sometimes has not:

> 'Indeed, the predicate of humanity is appropriable to Caesar in every moment of his existence, the predicate of triangularity is not appropriable at all, and the predicate of sickness is appropriable to some moments, and is not appropriable to others' [1, p. 50].

Vasiliev stresses that singular statements on the fact and singular statements on the concept obey different logical laws. The law of excluded middle is valid for the statements on the fact. For example, only one of two statements 'Ivan Ivanovich is drunk now' and 'Ivan Ivanovich is not drunk now' is true. Statements on the concept obey another law — the law of excluded forth. For example, one of three statements ‘Vega always shines’, ‘Vega never shines’, ‘Vega sometimes shines, and sometimes does not’ is true, and these three propositions are pairwise incompatible.

Vasiliev also supposes any singular statement on the concept to express the certain rule:

> ‘They describe time series as a rule, and the basic law for rules, the law of excluded forth acts for them’ [1, p. 51].

Thereby, Vasiliev in fact treats these statements as modal.

Vasiliev also considers the modal treatment of the propositions in his main paper ‘Imaginary (non-Aristotelian) logic’. Here he studies statements with universal, but not singular subjects.

In this paper Vasiliev proposed not only intensional but also modal interpretation of categorical statements containing in Imaginary logic:

> ‘If we take a concept as a subject of a statement, then any predicate refers to it either as 1) this predicate is necessary for the concept…, and we express this fact in affirmative statement about the concept…, or as 2) this predicate is impossible for the concept…, and we express this fact in negative statement about the concept…, or as 3) this predicate is compatible with the concept… The third case should be expressed in special accidental statement about the concept… This statement has its special copula different from the copulae of affirmative and negative statements’ [1, pp. 81–82].

The character of this copula is clear from another fragment where Vasiliev specifies the form of the accidental statements: ‘S possibly is and possibly is not P’ [1, p. 125].

Obviously, these modalities used by Vasiliev describe the type of predication, the mode of connection between the subject and the predicate, i.e. they are *de re* modalities.

Affirmative propositions are treated as containing modality of necessary inherence of a property to an individual, negative as containing modality of necessary lack (or impossibility for an individual to have a property), and indifferent as containing modality of contingency.

The formal explication of modal interpretation of Imaginary logic by means of special logic with *de re* modalities was proposed by V. Markin in [4]. The translation of formulas of the system **IL** into the language of logic for *de re* modalities was presented. It was demonstrated that the translations of all **IL** theorems are valid in this modal logic, while the translations of all theses rejected by Vasiliev are not valid here. Therefore, this modal interpretation is reasonable just for the main version of Imaginary logic. In contrast to 'intensional interpretation' it is not required to revise the set of its laws.

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The laws of reason and logic in Nikolai Vasiliev's system[1]

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abstract. The ideas of Russian logician Nikolai Vasiliev concerning the status of the law of contradiction are discussed in this article. The arguments presented in his article 'Logic and meta-logic' are deeply explored bringing to light the weakness of his philosophical theory. His 'imaginary' logic is a system that describes not the system of the laws of reason, but relations in which objects of some ontology stand to each other. Comparing the fundamental idea of Vasiliev to the classical concepts of reason brings us to a better understanding of the fact that philosophical intention of Vasiliev has been left unfulfilled.

Keywords: laws of reason, laws of logic, Nikolai Vasiliev's logic

The issue of the correlation between traditional logic and modern one is one of those which retain their controversial nature for many years. It is not surprising, because, as long as logic stays a philosophical science, it keeps asking itself about its origin and its subject. In this article, the issue of correlation of the laws of logic and those of reason is discussed being put within the context of ideas of a well-known Russian logician, Nikolay Vasiliev. In Vasiliev's works the investigation of the nature of logical laws acts as a foundation for building the systems of non-classical logic. The intrinsic connection between solving a philosophical problem of the nature of logical laws and the possibility of building a new logic seemes obvious to Vasiliev. Although it does not seem that obvious to us anymore, Vasiliev's theories keep attracting vivid interest whenever the nature of the logical is discussed.

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My interest in Vasiliev's logical theory has been aroused by the fact that some of my colleges have recently developed a deep interest in his theories and seem to have found in his writings a range of fascinating logical ideas. The fact of these ideas being fertile does not cause any doubts. Though, it seems to me, that it is not so much the ideas themselves that are fertile but the ingenious and enthusiastic incentives given to them by Vasiliev.

In this article I am going to focus on his work *Logic and metalogic*, where he provides a foundation for a statement which is both crucial and fundamental for his 'Imaginary logic'. According to this statement: 'Thinking may change, but it is not everything there that is changeable' and also 'there are some absolute logical truths but it is not all the truths of logic that are absolute' [1, p. 331(96)].

This statement is grounded by Vasiliev with several, partly interconnected arguments, which I intend to analyze critically. Let's briefly recreate the way of his argumentation.

The first argument points out the existence of analytical and synthetic truths, whereof the first are necessary while the second are not. Do all the laws of logic have analytical character? This is the question which Vasiliev raises drawing on the similarity of the laws of logic and those of geometry. Vasiliev keeps stressing the parallelism existing between his imaginary logic and the imaginary logic of Lobachevsky.

The second argument is based on this similarity. Vasiliev argues that, if the 5^{th} postulate of Euclid is independent from the others and so maybe substituted by some other argument without any contradiction arising, then the similar condition should lead to the similar consequence for the law of contradiction. In other words, having got rid of the law of contradiction, that is allowed contradiction as logically possible, we, in case that it is not dependant on other logical laws, should not have any non-contradictory results. The assumption seems paradoxical because, while, on the one hand, a contradiction is allowed, on the other hand, we expect it not to lead to a contradiction, that is, to this very thing which has just been allowed. In fact, Vasiliev, in his own, indistinct way, presupposes both inside and outside ways of considering logical reasoning. These ways today we would call the levels of meta-language and object-language. Allowing a contradiction as possible, in Vasiliev's theory, means that predication may turn out both true and false at the same time, but the fact itself either takes place or does not. The further developing of this idea leads Vasiliev to two kinds of negation, whereas the logicians following him this way have been brought to constructing many-valued and paraconsistent systems.

The third argument is an ontological one. Vasiliev points at the world of fulfilled contradiction created in the systems of Nicolaus Cusanus and Hegel. 'When they were thinking contradiction as existent and actual, were not they thinking logically?', Vasiliev asks. Being carried away, as it were, by this argument, he speaks further on about the Earthly logic of the law of contradiction setting it against the logic of some remote corner of our Universe, where contradictory things may exist. In that remote place, he argues, reason would become accustomed to the triple division of propositions into the true, the false, and those having the third meaning, and would act accordingly. It should be mentioned again that the assessments of such a reason would stay double-semantic, that is, noticing, or grasping a contradiction as existing, such a reason would not be able to assert the existence of a contradiction along with its absence. Hence, Vasiliev derives the dependence of some logical laws on the conditions of experience, e.g. he tries to provide a foundation for their empirical nature. Changing of empirical sphere leads then to changing of their laws. In this he follows Kant, who divided logic into the general (pure) one and the applied, although, according to Kant, the latter is the sphere where the laws of pure logic are applied to specific experience.

Empirical nature of some logical laws is founded by two following lines of argumentation.

Lets' follow the first one. A criterion of any law's empirical character consists in its ability to be eliminated. It means that a law may be substituted by another so as to retain its non-contradictory nature. It should also be said that Vasiliev considers the empirical as a criterion for being beyond logic and rationality. The empirical character of the law of contradiction is provided with grounds by the very fact that imaginary logic does exist — the logic where contradictory predicating is just one of the ways to predicate, although the logic of propositions is still the classical one.

The second proof of the empirical character of the law of contradiction is based on Kant's formula of this law. According to this formula 'There is no object which predicate can be contradictory to the object itself'. Vasiliev founds his law of contradiction on incompatibility of objects' qualities, i.e. on it being impossible to predicate more than one quality simultaneously, which, according to Vasiliev, creates a basis for negation. Incompatibility of qualities, argues Vasiliev, is an empirical condition.

The forth argument brought forward by Vasiliev actually is the developing of the third. Empirical logic is claimed to be something which is created in the process of 'life and struggle' and serves as 'a live organism, a means of struggle, and a reflection of both environment and a man'.

If we forget for a while who Vasiliev is, and make an attempt at an objective investigation of his arguments, then we can't but admit that they are hopeless.

His referring to the imaginary logic of Lobachevsky is hardly suitable. As it is well known, Lobachevsky first tried to prove the 5^{th} postulate of Euclid expecting that the supposition of it being negated would lead to a contradiction. After he had found that it hadn't been the case, and having discovered a new geometry, he tried to find a model for this new science. Later on, Beltramy managed to do this. As for the law of contradiction, as Vasiliev sees it, the case is completely different. When he speaks about predication, he postulates its triple character straight away, at the same time, strongly emphasizing the retaining of classical laws on meta-level. Thus not only the system of postulates, but the very principle of the theory functioning is being changed. While with negation of the fifth postulate of Euclid nothing has happened either to the ways of constructing conclusions, or to the character of semantically assessing geometrical statements, negation of the law of contradiction by Vasiliev turns out to belong to an 'inner' sphere of some phenomena, namely, to the phenomena of predication, but not to logic itself. Predication, in its turn, may be three-valued. Vasiliev does not examine whether this triple character is compatible with other principles of classical logic. The result of such an examination is easily predictable, but Vasiliev does not comment on changing the meaning of the thesis he defends.

Just as far from Vasiliev's theoretical efforts stays intuitionism, which, as we remember, rejects the law of the excluded third as a result of having adopted semantic attitudes different from those of classical logic. Here, we may see a clear case of applying a method of investigation correlated with that of Lobachevsky. Vasiliev's pointing out the synthetic character of the statement 'The sum of a triangle's angles equals two square angles' corresponds with Kant's views, but it has nothing to do with the laws of logic. Moreover, if we add to this formula a concretization 'on the surface with a zero curve', then we'll have a still synthetic proposition, but of an apodictic character, which would reflect, according to Kant, a result of pure contemplation. It is rather risky to refer to a distinction between analytical and synthetic in Kant's theory for the sake of purely logical investigation. In his *Critique of Pure Reason*, synthetic character is asserted for elementary arithmetical equations, while general laws, such as commutativeness of adding, are claimed to be analytical. To criticize Kant's concepts of analytical and synthetic is a common place thing. In fact, according to Kant, when we are thinking the sum $5+7$, we are not thinking the number '12'. But are we thinking '2' anyhow differently from the sum '$1+1$'? If, when thinking $a+b$, we, according to Kant, are thinking $b+a$, does it mean then, that when thinking $(a+b)(c+d)$, we actually are thinking $ac+ad+bd+bc$ as well? Concepts of analytical and synthetic are interpreted by modern logic rather along Leibniz's way of drawing a distinction between truths of reason and truths of fact. Then, deduction of a logical law from a system of postulates of any concrete logical system becomes a regulative of analytics of judgment. Kant understands logic as a science of 'necessary laws of understanding and of reason in general, or what is one and the same, of the mere form of thought as such' [2, p. 528(320)]. He agrees with Leibniz in this, and it shows also in his rejecting any possible psychological roots of logic. Kant especially emphasizes, that logic is a rule for any application of either the reason or understanding. Moreover, it is such a rule, which is uncovered in the process of investigating of how understanding carries out its cognitive activities. Thus, to question the thing, which is defined as a law of logic, would mean, according to Kant, to deprive both reason and understanding of their capability to act. In the case of Vasiliev's idea, and following Kant's way of thinking, we would have to bring forward a hypothesis, that thinking is possible without the law of contradiction, that is, that such a law is not logical. In this case, a difficulty arises: we should decide whether it is with using the law of contradiction or without it that we should discuss the results of accepting such hypothesis. The example of Lobachevsky here cannot serve us a guiding point because of the difference in the subject of investigation, which has been already pointed out.

Ontological arguments demonstrated by Vasiliev are the weakest and most unconvincing. The concept 'contradiction' has many meanings so that it is not advisable to mix up the meaning usually assigned to the term in logic with the meaning it acquires in philosophical theory, where contradiction may be understood as the presence of opposing tendencies in an object or a phenomenon. His explaining the origin of the law of contradiction through referring to 'life and struggle' just adds some not very sophisticated phychologism to this terminological confusion. To say something in his defense, we can remember here a lot of authors who exploit the concept of contradiction as a metaphor not caring or caring too little about either logical, or philosophical precision.[2]

Finally, the proofs of empirical nature of some logic laws fall apart. The first one makes a logical circle as it explains the empirical character of the law of contradiction through postulating the possibility to build a logic without this law altogether. But, the first step towards building of such a logic consists in claiming the law of contradiction as not functioning in predicating. And there are no attempts made, as we've seen, to consider whether the law is compatible with other principles or not.

The second proof is based on the actually wrong understanding by Vasiliev of Kant's formula and on his changing the thesis, as it were. Vasiliev takes no notice of Kant's example 'No uneducated

[2] Alen Badiou gives a good example of how simply logical interpretations of the law of contradiction may be used for producing a postmodern text. See: [3].

person is educated' where we actually deal with contradictory predicates. He keeps saying that 'white' and 'black' are incompatible instead of talking about 'black' and 'not-black'. Thus, instead of considering the impossibility to negate actually a predicate an object possesses, Vasiliev is considering the predicates which are not logically related in this way. The law of contradiction is understood by Kant as a general formal condition of knowledge agreeing with itself, as condition sine qua non, which comes before the question of truth is raised [2, pp. 558—559(358)]. Moreover, formality here means independence from any content, and so much so, that counter-posing of 'black' and 'white' contradicts Kant's interpretation, where it is only for logical counter-position of 'A' and 'not-A' that the place is found. Kant thinks that understanding and reason never deviate from conforming to necessary for them logical laws, whereas all the mistakes and false concepts come from the actions of sensual cognition: we confuse our subjective foundations for judgments with objective ones, truth — with its appearance. The cause and origin of all mistakes and false ideas, as Kant put it, lies in the precipitation and rashness with which we use our understanding [2, pp. 560–561(361)]. This remark by Kant is interesting in that it does not allow of any alternatives to the existent logical laws. Reason and understanding of a madman or of a primitive person act in accordance with the same laws, which control and organize the most perfect mind in the world.

That is why Vasiliev tries to avoid polemics with the tradition of classical rationalism. Also, it is hardly suitable to mention Leibniz in this context because the latter, when describing the law of contradiction, talks about the impossibility to simultaneously predicate something to an object and to negate this predication; and also, he explains the impossibility for a statement to be true and to be negated at the same time: '... any proposition (be it either an affirmative or a negative statement) may be either true, or false; where, if a statement is true, then its negation is false; if a negation is true, then it is the affirmation that is false. If the truth of something is negated, then (obviously) this something is false; whereas, if something is negated as false, then it is true' [4, pp. 299(138)]. In his *Theodicy* Leibniz puts it this way: 'of two contradictory propositions one is necessary true, and the other is false' [5, §44]. In his *Monadology* Leibniz calls the law of contradiction a 'great principle', due to which we 'think as false that, which in itself contains a contradiction' [6, §31]. For Leibniz, as well as for Descartes, the laws of logic belong to those which are absolutely necessary and indispensable for reason, to the eternal truths, that are uncovered by reason without any other experience, but the experience of thinking itself, and are innate to thinking. Certainly, as Leibniz observes, it is not anyone that can uncover the truths innate to one's consciousness. This requires certain efforts, but if these efforts done, the result is achieved which is apodictic [7, Book 1, Ch. 2, §12]. Here Leibniz is more cautious than in the passages cited above. Uncovering of some absolute truths requires, certainly, not any data of experience, but carrying out intensive thinking activity in connection with experience. We can trace here the influence of Plato's theory of knowledge as remembering, which is related in his dialogue *Menon.* This theory is interesting in its drawing a line between obvious truths, which are easy to grasp and the truths, which can be uncovered and understood only with the combined efforts of both a teacher and a student. In epistemological logic such a division corresponds to actual and potential knowledge. To the latter the knowledge of truth of arithmetical equation belongs, in which two serious polynomials are on the left and the right side: to get an assuredness in an equation being true it is necessary to do the required calculations correctly. Plato, as well as Leibniz, would not place the law of contradiction among truths of this kind, because its application is a necessary condition for all thinking operations. This means that the law of contradiction belongs to ever actual knowledge.

All in all, Vasiliev follows Mill's interpretation of the law of contradiction without escaping Mill's lapse in argumentation pointed out by E. Husserl in his *Logical Investigations.* As we know, Mill claims the law to be empirical when there are no grounds for it to be so. In other words, instead of scientific empiricism, the results of which are not to be ignored, we are faced with the simple arguments of common sense, which, in their turn, disguise the metaphysical premises, that hold experience as the only source of cognitive forms.

So, as it happens, the whole article *Logic and metalogic* should be considered as a failure. The intuition underlying this work is of a more serious nature though.

To correctly assess possibilities of applications for Vasiliev's ideas it is worthwhile to remember the classical concept of the laws of reason.

We cannot consider as trivial the statement which claims that the laws of logic are the laws of reason as such. It is so not because today we have a number of various 'logics', which someone may consider to be alternatives to each other; and also, not because Vasiliev apparently was the first to open the door for these logics to emerge.

Classical concept of laws of reason goes down to Aristotle and Leibniz. Hegel is undoubtedly among its supporters. I'm going to present here some crucial points of this concept along with the criticism aimed at them.

First of all, Leibniz holds the laws of logic to be those of reason as well because they are discovered by reason itself as being self-evident. Certainly, under scathing attacks of modern criticism aimed at the theory of cognition the self-evidence of Leibniz and Descartes may be shattered. Both these philosophers were helped and supported by the natural light of reason given to men by God and in its light showing the self-evident. Yet, our intuition is similar to that natural light only in that its effect is perceived by us as something not dependant on our empirical self, but rather as a result of functioning of transcendental grounds of our thinking. To some extent, we may pass it by without paying much attention. What intuition deals with is not as important as the way it functions. In other words, the laws of reason are still obvious for us, though only to that degree which is available for us.

Secondly, this degree of evidence (obviousness) is certainly much weaker than the one Leibniz talks about. Leibniz's obviousness was issuing directly from God, who had given us this ability to investigate truth, while ours is pre-determined empirically. This is caused by the fact that, although these transcendental grounds of obviousness have been formed out of the sphere of our conscious activity, still it has occurred in the process of our forming as psychologically wholesome entities. It is here that the theoretical foundation lies for claiming the laws of logic not the laws of a universal reason, but the laws of a concrete, specific reason — the reason of a specific person, specific period and so on.

Thirdly, Leibniz had given quite a successful development to Plato's and Aristotle's teaching of ideas. Leibniz claims that God can neither destroy nor reject an essence, though all the essences had been contained in God's reason before even the universe was created. God of Leibniz as super-reason appears to be over laden with all its creations — both essences and laws. If the hypothesis of God is to be excluded, then there are nor essences neither laws. There are only concepts left created by our concrete reason in the process of reasoning. These concepts are chosen by us quite arbitrarily according to the order and way which are at our disposal here and now. That is why, hypothetically, any other way of creating concepts and working with them is legal, as it were.

To what extent may the alternatives to actually existent reason be represented by this reason itself? This is the question raised by Vasiliev, and he gives the only possible answer — to no extent and under no conditions. But, why is this so?

Independently from its source and from its either empirical or historical state, reason always creates its laws. Reason does not have this possibility to pick up laws while considering different options, but it is always pre-determined by its transcendental ground. Not having a possibility to act differently than it does, reason is deprived of any possibility to contemplate an alternative model for itself. If we want to present any way of reason's functioning as a way possible for it to realize, then actually existing reason just won't be able to assess this new way because it won't be able to see itself acting along the presented new scheme. This means that there is only one actual logic which expresses reason's laws, and everything deviating from it just can not be logic because doesn't come up to the requirements set up by its definition. This is the way it all looks as seen in the perspective of the classical teaching.

Yet, Vasiliev, when talking about meta-logic, has in mind exactly logic, because, as he turns to empirical logics, by which he means some 'imaginary' logics, he uncovers a sphere of formal description of arbitrary objects, which, just because being imaginary, can not be logical. Vasiliev's intuition is directed at a play with the axiomatic of an arbitrary theory, which, due to some misunderstanding, has happened to be applied to the sphere completely unsuitable for it, — the sphere of logic. The value of Vasiliev's approach for modern logic lies in the fact that it chooses the logic as a universal instrument for analyzing any system of relations. Yet, we should not forget that such an application of logic is tied up not on its nature, but on the fact that the form of logic makes it easier to grasp and understand the very principle of systematic organization of arbitrary objects relations. Logic as a *canon* of reason appears to be very attractive for this role of an instrument of investigation of arbitrary relations. Thus, Vasiliev's approach fulfills those intentions of the 19^{th} century's philosophical tradition which considered logic as an *organon* and tools of investigation and was expecting quite a lot from logic's merge with empirical sciences.

The questions remains open, whether any system of many-value assignment is expressible by means of any meta-language. As it has been already mentioned, Vasiliev retains meta-language as classical so that, having agreed with the presented interpretation of his ideas, we obtain either paraconsistent, or three-value logic, which is to be interpreted in two-value meta-language. Thereby, all the philosophical pathos of building a foundation for an 'imaginary logic', as well as all the efforts to eliminate the law of contradiction, just vanish, or turn out as vain: it turns out that we classically argue about a local non-classical application of reason, which may be interpreted in plenty of other ways, which would inevitably bring us back to classical reason. And why, we may ask, should we not try to realize fully Vasiliev's idea, that is to reason non-classically in meta-level? For if, according to Vasiliev, the law of contradiction is not an empirical one, then there is only one ground for using it in meta-level: it is not an 'imaginary' reality, but an actual one which constitutes the experience that makes us follow this law. It is clear why the reasoning about real world, as well as this reasoning's descriptions in meta-language, should be regulated by the law of contradiction. But turning to 'imaginary' world, where the law of contradiction does not function, we, obviously, leave the sphere of actual, so that using the law of contradiction in meta-level lose any grounds or justifications. In fact, turning up, as it were, in an 'imaginary' world we should have conformed to its reality, and rejected the law of contradiction in meta-level as well. Although we physically stay in the real world, when creating an 'imaginary' logic, we get transferred to its world in our thoughts, and classical meta-language remains just a means of correlating of the two worlds. The proper realisation of Vasiliev's 'imaginary' logic requires that it is built and interpreted by means of 'imaginary' meta-logic.

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Technical systems in logic: questions of formalization and automatic verification

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abstract. In the paper technical systems with counters are considered as logical models. The questions of formalization in temporal logic and automatic analysis via computational tree transformations are discussed.

Keywords: temporal logic, verification, formalization, proof constructing

1 Introduction

Some classes of modal logic such as CTL (computational tree logic) and LTL (linear temporal logic) are now rather widely used in computer science for the task of verifying finite automata [2]. To be verified, an automaton must be rewritten in terms of the corresponding logic, and then some of proof-constructing techniques, automatic or interactive, are applied to find whether the automaton performs actions it is supposed to perform (and no other). If the formalization is sound this procedure must reveal all the possible errors that can appear in the automaton. The techniques of formalization and analysis of finite automata and automaton programs via temporal logic are widely known as model checking approach. But the sense of temporal logics isn't restricted by only automata or program systems: they can describe any lasting-in-time process, every moment of which can be described by the classical logic. Though some of these processes are very hard to be described (e.g. due to possibility of infinite branching), others have very simple structure, so temporal logics can be implemented to them even easier than to program systems. In particular, it is possible to expand field of applications of the classes of logic to parametrized electromechanical technical systems. The linear temporal logic was successfully used to perform an analysis of new reagent-dozing mechanism [11]. Despite the fact that results of the analysis were confirmed by an experiment, some problems were discovered in this approach. The hardest one was to find proper proof-constructing instruments.

The matter is that electromechanical systems like the reagent-dozing mechanism are parametrized and their features to be checked essentially depend on these parameters. So in the most general case induction scheme must be included to make the analysis possible (and it was used to perform the analysis in [11]). Moreover, even one-counter automata representation in modal logics can lead to undecidability [3]. But in every specific case we need only very restricted version of the induction. So neither model checking instruments nor interactive provers fit the task perfectly: the former are too weak in some aspects (no induction) and too complex in others (allowing very complicated automaton structures); the latter are too strong and therefore demand human decisions. Our suggestion was to try a universal technique of program transformation on this problem, and the first results of this approach have been received.

The technique is called supercompilation and was developed since seventies of the last century (starting with the works of V.F. Turchin) [15]. Its essence is unfolding a computational tree of a partly specialized program together with folding it back to a program graph. The term 'partly specialized' means that some input parameters of a program can be known (and some can remain unknown). The process of unfolding resembles the one that is performed in model checking on automata. In general, supercompilers have no domain-specific transformations to optimize complex finite structures so they cannot compete with corresponding model-checkers but the technique of supercompilation was successfully applied to verify simple parametrized protocols [9, 1] and in some of these applications showed itself even more powerful than the corresponding domain-specific programs. Despite the supercompilation wasn't presented as a proof-transforming technique, it has very deep interconnections with logic, and its applications to the temporal logics CTL and LTL lead to some appealing effects.

Therefore we try to construct a 'double permutation' of applying modal logic to technical systems and computer science to modal logic. The results show that all these domains have very much in common, but still a lot of work remain on this way.

2 Infinite state technical system in LTL

2.1 General features

Suppose that a standard circuitry is enriched by some parametrized elements, such as switching boards, and the system is not only to satisfy some qualitative features but has some quantitative ones. Then it may happen that in states of the system never repeat themselves in time. For example, if we have a reagent-dozing system that spreads a reagent of a volume V_r after a volume $K * V_w$ of water passes through a pipe, then in most moments of time the concentration of the reagent in the water reservoir is unique (it is expressed as a formula $\frac{V_r}{V_w} * \frac{T}{T*K+R}$, where T is the total number of full cycles of the mechanism work and R is the number of water volumes V_w that passed through the pipe after the last reagent spread ($R < K$)). In such cases it is reasonable to examine only changes of system quantitative features but not the features themselves. Because of the discrete nature of the parametrized elements that are included in the scheme, these quantitative requirements can be approximated by rational numbers. Because of finiteness of the scheme the satisfiable requirements of state change must belong to some finite (though maybe very large) set. Since parameters of the parametrized elements can also change, the set of the requirements, though remains finite, becomes even larger.

So the following features of the system are guaranteed:

1. Non-constant finite number of elements;
2. Finite set of different states (without considering quantitative aspects);
3. Finite set of changes of quantitative characteristics of the system on a fixed state.

And the following features are to be checked:

1. Uniqueness of successor state. When human intrusion is forbidden, it is essential to know that the system works without any unpredictable indeterminacy.

2. Existence of successor state. If some reachable state never changes, the system will halt.

3. Repeatability of all significant actions.

4. Return to initial state.

These features make us sure that the system infinitely repeats the loop between two initial states.

The whole description of the system somehow resembles the one of the modular arithmetic. The similarity becomes even greater in practice because of the round construction of switching boards that allows a stepping mechanism to move only forward (and the last switch is followed by the first).

In the logical model of the system we describe two classes of axioms: axioms of stability (forbidding actions to be done without a trigger) and axioms of change (describing possible one-step changes of conditions). For every action possible in the system at least two axioms appear, and one more axiom must be included to describe initial conditions. If the system of axioms is full, the feature of linearity (the uniqueness of successors) can be checked by checking the fact that antecedents in every two axioms of change are disjoint. To prove the features of existence and repeatability it might be useful to find additional invariants of the system. After that the last feature can be checked constructively by building a single iteration starting and ending by the initial state. In every specific case, when all parameters of the scheme are known, it can be done in straightforward way.

2.2 LTL formalization

We will use the full language of LTL with the modalities **X**, **G**, **F**, **U**. Though in the first formalization the modality **U** was used in the model axiomatization, now we try to avoid it to make the model better from the point of view of computer science.

As an example of technical system let us consider a simple dozing mechanism which consists of the single stepping switch, the single pump and the single flow indicator. Every time the flow indicator turns the stepping switch does one step. One of the contacts of the switching network (which can be chosen arbitrarily while mounting the scheme) is connected with the pump. When the stepping switch reaches the contact (let us call it K) the pump does one move and closes the chain that returns the stepping switch to the first position. This sequence of actions repeats itself potentially forever. It is less complex than the scheme in [11]: now we have only one stepping switch instead of the two — but the analogue of our simplified scheme is also used in practice [16]. So in fact we need to verify some special kind of one-counter (not necessarily determined) automaton in LTL — the problem that is known to be PSPACE-hard in general and that becomes undecidable after adding one more counter [3].

The scheme is parametrized with the two parameters: the dimension of switching network N and the number of the contact K that activates the pump. So we need to use not only single propositional variables but also the array of switch states on the switching network, and not only single axioms but also axiom schemes corresponding to different states of stepping switch. More natural approach is to consider not only boolean variables but also a variable of the type $\mathbb{Z}_N$, and this approach is valid due to the two features of stepping switch construction:

1. The stepping switch is always connected to one contact on the switching network;

2. This contact is unique.

The normal work of the mechanism assumes these two features. But if we want to study disruptions then we must use the model with array of booleans since we assume short circuits to happen on the switching network. We now consider most generic case, and turn to the usage of arithmetic in the program model.

The model variables are the following.

1. R is the state of the indicator switch;

2. C is the state of the switch returning the stepping switch to the initial state;

3. P is the state of pump switch;

4. T is the state of additional switch controlling the pump;

5. W_i is the state of the stepping switch (N is the total number of contacts, so $i \leq N$).

The model axioms are separated in two sets. The first is called axioms of change and describes how a current state can affect its successor. Construction of these axioms can be done straightly from the description of the mechanism work.

1. $R \wedge W_i \Rightarrow \mathbf{X}(\neg R \wedge \neg W_i \wedge W_{i+1})$ (the axiom scheme for all i, $i \geq 1 \wedge i < N$).

2. $R \wedge W_N \Rightarrow \mathbf{X}(\neg R \wedge \neg W_N \wedge W_1)$

3. $\neg R \wedge T \wedge \neg W_K \Rightarrow \mathbf{X}R$. This axiom, when implemented, means that the volume of water V_w have passed through the pipe.

4. $P \Rightarrow \mathbf{X}(\neg P \wedge \neg T)$

5. $\neg P \wedge W_K \wedge T \Rightarrow \mathbf{X}P$. This axiom, when implemented, means that the volume of reagent V_r have been spread into the pipe.

6. $\neg T \wedge W_1 \wedge C \Rightarrow \mathbf{X}(\neg C)$

7. $\neg C \wedge W_i \Rightarrow \mathbf{X}(C \wedge \neg W_i \wedge W_{i+1})$ (the axiom scheme for all i, $i \geq 1 \wedge i < N$)

8. $\neg C \wedge W_N \wedge \neg T \Rightarrow \mathbf{X}(C \wedge T \wedge \neg W_N \wedge W_1)$

The second set of axioms is called axioms of stability and forbids switches to change their conditions arbitrarily. In general it is not obvious how to construct these axioms because they are hidden inside a scheme and not presented explicitly in its description. This fact may lead to mistakes in a scheme (for example, in the mechanism tested in [11] some uncertainty appeared exactly because of the absence of the axiom that forbids the indicator switch to change state during the work of pump).

1. $W_i \Rightarrow (C \wedge \neg R \Rightarrow \mathbf{X}W_i)$ (the axiom scheme for all i, $i \geq 1 \wedge i \leq N$)

2. $\neg W_i \Rightarrow (\neg W_{i-1} \vee C \wedge \neg R \Rightarrow \mathbf{X}(\neg W_i))$ (the axiom scheme for all i, $i > 1 \wedge i < N$)

3. $\neg W_1 \Rightarrow (\neg W_N \vee C \wedge \neg R \Rightarrow \mathbf{X}(\neg W_1))$

4. $T \Rightarrow (\neg P \Rightarrow \mathbf{X}T)$

5. $\neg T \Rightarrow (C \vee \neg W_N \Rightarrow \mathbf{X}(\neg T))$

6. $\neg P \Rightarrow (\neg T \vee \neg W_K \Rightarrow \mathbf{X}(\neg P))$

7. $\neg R \Rightarrow (\neg T \vee W_K \Rightarrow \mathbf{X}(\neg R))$

8. $C \Rightarrow (T \vee W_1 \Rightarrow \mathbf{X}C)$

All these axioms describe the dependence of variables' values in a successor state on their values in the current one. And the last axiom remains to be introduced — the axiom of initial state: $W_1 \wedge \neg R \wedge \neg P \wedge T \wedge C \wedge \forall i(i > 1 \Rightarrow \neg W_i)$.

Now we must formalize the conditions to be checked.

1. Uniqueness of successor state. Only one axiom of change can be implemented at every moment of time.

2. Existence of successor state. At least one of the axioms of change can be implemented at every moment.

3. Repeatability of all significant actions.

   $\forall i(R \wedge W_i \Rightarrow \mathbf{XF}(R \wedge W_i)$

   ...

   $\forall i(\neg C \wedge W_i \Rightarrow \mathbf{XF}(\neg C \wedge W_i)$

4. Return to initial state.

   $W_1 \wedge \neg R \wedge \neg P \wedge T \wedge C \wedge \forall i(i > 1 \Rightarrow \neg W_i) \Rightarrow \mathbf{XF}(W_1 \wedge \neg R \wedge \neg P \wedge T \wedge C \wedge \forall i(i > 1 \Rightarrow \neg W_i))$.

The only task is now to present a fast and efficient method of discovering the features of the formalized system. In [11] it was done by hand though there are some techniques in program transformation that can be (and have been) successfully implemented to perform this work. One of these techniques is the supercompilation which is typically used in optimization and analysis of functional and imperative programs [4, 7], but last years it was also implemented to verification tasks [9, 1]. It was noticed that this technique works well on simple parametrized systems of various natures [8].

3 Proving in LTL via supercompilation

The definition of supercompilation can be found in [14]. Saying informally, it consists of the following techniques:

1. Unfolding a computational tree.
2. Folding some branches of the computational tree by means of generalization.
3. Extracting a residual program from the folded graph.

The first technique unfolds computational tree of a program, by step-by-step driving: implementing its rules on computational states until any functional calls disappear. It is exactly the same technique that is used in model checking while unfolding infinite graph of states of an automaton. This resemblance implied the idea that supercompilation may be used for modal logics.

The second technique belongs to the supercompilation itself and is the heart of the method. Some computational branches can be infinite so it is necessary to stop driving them without reaching a final state. For this need the following technique is developed: if some computational branch is considered to be ‘dangerous’ then driving on it halts. The notion of being ‘dangerous’ can vary in different supercompilers (e.g. it may mean ‘to be too long’ or ‘to repeat itself’). After discovery of a dangerous state a supercompiler folds the computational branch into a loop using generalization. Generalization unifies a latter computational state of the branch with some former state on it that most resembles the latter state (the notion of ‘resemblance’ can also vary in different supercompilers).

After all computational branches of a program are computed into final states or folded, a residual program is constructed from this folded graph. This step isn't discussed in this paper because it is very dependent from a programming language of a target program.

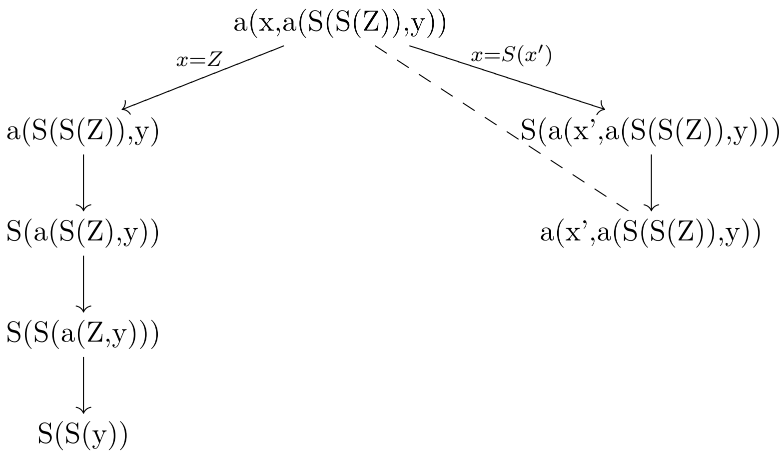
Let us illustrate the process of supercompilation on a simple example from Peano arithmetic.

Example 1.

There is a recursive definition of addition.

```
a(Z,y)=y;
a(S(x),y)=S(a(x,y));
```

Let us supercompile the call `a(x, a(S(S(Z)),y))`.



Every step is an application of a definition or case analysis. The supercompilation lasts until there are no functional calls in a node or the node repeats its predecessor modulo renaming.

Thus, the residual recursive definition corresponding to the call `a(x, a(S(S(Z)),y))` looks as

```
f(Z,y)=S(S(y));
f(S(x),y)=S(f(x,y)).
```

The second addition is already evaluated by the supercompiler so it disappears in the residual program.

In the terms of logic, supercompilation technique rewrites an initial proof for some special case of a theorem (in the example the theorem was an existence of the sum of every two naturals) and then does some obvious logical transformations to receive the proof that isn't more complex than the initial one (it is desirable to make it less complex) but is equivalent to it by means of the set of realizations. There are some restrictions of input proofs: positive supercompilation allows no $\neg$ in disjunctions; perfect supercompilation allows introducing constraints, so it permits $\neg$ in case analysis.

The set of the allowed logical transformations of a proof is very limited and depends on supercompilation technique. For example, the rule $A \wedge A \Leftrightarrow A$ is equivalent to so-called msg (most specific generalization) sharing mechanism and is used in positive supercompilers [13]; the rule $A \vee A \Leftrightarrow A$ is used in the supercompiler SCP4 [10]. The last uses even more 'unsafe' transformations such as $\neg\neg A \Rightarrow A$ that allow to highly optimize very complex programs but in some cases changes their semantics (e.g. transform infinite proofs to finite ones).

Example 2. Consider a recursive function on natural numbers:

```
g(Z)=Z;
g(S(x))=g(h(x,Z));
h(Z,y)=y;
h(S(x),y)=h(x,2^y);
```

`g(x)` is non-terminating for $x > 1$ but in an unsafe supercompiler the observation that all the computations cannot end by the term other than `Z` (so $\forall x(\neg\neg(g(x) = Z))$) may lead to semantics-changing optimization replacing the definition of `g` by a constant zero.

The supercompilation technique doesn't fit well for every task on automatic analysis and has some subtle points such as semantic-changing transformations. But it was successfully used for verification of cash-coherent protocols [9] and communication protocols [1] in cryptography. We decided to try it also on verifying the model of a simple dozer that is described in the previous section.

As input language we chose Refal which is close by semantics to Markov algorithms on strings. It was done because Refal is the input language of the general-purpose supercompiler SCP4 [10]. The convenience of this supercompiler is that it presents perfect information propagation in conditions (allows $\neg$) so it is possible to use not only boolean variables (for which perfect information propagation coincides with positive), but also naturals. SCP4 is not semantics-preserving in case of non-terminating target programs, but if the target program always terminates, this supercompiler preserves its semantics, so the problem is insignificant.

Representation of the model variables repeats the one in the previous section, but we chose unary natural number to represent the state of stepping switch. Representation of the model axioms in programming language was straightforward: we divided data visible by the program on two parts, a current state and a next state. The initial state is completely determined. All the axioms describe the influence of the former to the latter. If after implementing all the axioms some uncertainty remains in the next state, then this uncertainty is represented as free variables which can be replaced by any value. The axioms of change that correspond to changing the state of system (counting water flow or pumping the reagent) increase corresponding variables: one of them, `e.R`, is the number of volumes V_w that passed through a pipe; another, `e.P`, is the number of volumes V_r that were added to the water flow. The aim is to prove that with any values of free variables the program returns to the initial state and computes the certain pair, `e.R, e.P`, and then to compare $\frac{\texttt{e.P}}{\texttt{e.R}}$ to the technical requirements (it must be equal to $\frac{1}{K-1}$). The supercompilation does this task for any constant N and K. If $K \neq 1$ and $K \leq N$, then the target program is optimized to `'FF'` (that means that $\neg(\texttt{e.R} \neq K-1) \wedge \neg(\texttt{e.P} \neq 1)$). If $K > N$, then no optimization is performed, so the program has indeterminacy and represents the erroneous work of the mechanism. This situation simply means that there is no contact on the switching network that is connected with the pump. The most interesting fact is that for $K = 1$ program isn't optimized to constants, so indeterminacy appears and the system becomes inconsistent. This is an analogue of the error that was found in [11] (and, as was shown experimentally, really can occur in the system). By changing one or another class of axioms we can also see how the system behaves when some physical disruption happens.

But to check all practical cases by SCP4, it is necessary to write simple auxiliary program that lists all of them in the input SCP4 file. This inconvenience appears because modern supercompilers are very careful in transforming programs (or proofs) not to change their semantics (or admit very slight changes) and permit only very few logical transformations. It is supposed that strengthened techniques like distillation [5] can do more complicated transformations like merging several induction proofs in a one so can be able to avoid these restrictions.

4 Conclusion

Our consideration of mechanical models together with logic and computer science led to revealing a deep connection between all these domains. First, mechanical models are easily described via temporal logic with counters, through introducing two classes of axioms that permit and forbid to change condition of an element. Second, the axioms of temporal logic with counters can be easily rewritten as functions in a programming language strong enough to express negation. Due to the features of quantitative restrictions that can appear in such systems it is possible to represent the conditions of consistency of the scheme as well. And third, it is possible to implement non-domain-specific program transformers to do the automatic analysis of the model. The essence of the last fact is that we must only formulate model axioms and the conditions of correctness and incorrectness of the scheme, and then it is possible to use not proof-constructing, but only proof-transforming technique to verify every special case of the scheme.

This may be the small step to observing the pure theoretical results of reverse mathematics in practice. It is known that folding computational trees only along their branches in supercompilation leads only to linear speedups of residual program [6] (so only $I\Delta_0$-formulas may be eliminated). But embeddings of tree embeddings lead to some non-linear program speedups very close to ones performed by induction [6]. The fact is not surprising from the point of view of logic, but remains very subtle and obscure from the point of view of practical program transformations. Establishing understanding between these two points of view seems to be very fruitful, yet unexplored, domain.

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Appendix. Refal representation of the model

```
$ENTRY GO {s.1 s.2 s.3 s.4 (e.1) = <Launch (s.1 s.2 s.3
s.4)(e.1) (<I 'III'>) (<I 'II'>) (()()('I')'F' 'F' 'T'
'T')>;}
```

`Launch` **checking the consistency condition and applying all axioms*

```
  {(s.1 s.2 s.3 s.4)(e.1)('I' e.N)('I'e.K)
(('I'e.KK)('I'e.X)('I') 'F' 'F' 'T' 'T') = <Neq
(e.K)'I'e.KK> <Neq ('I'e.X)'I'>;
  (s.1 s.2 s.3 s.4)(e.1)(e.N)(e.K)((e.R)(e.P)(e.W) s.R
s.P s.C s.T)=<Launch (s.1 s.2 s.3 s.4)(e.1) <Next <AxiMW
<AxiMP <AxiMR <AxiMC <AxiSC <AxiMT <AxiWStab <AxiTStab
<AxiPStab <AxiRStab <AxiCStab (e.N)(e.K)((e.R)(e.P)(e.W)
s.R s.P s.C s.T) (e.R)(e.P)(e.1) s.1 s.2 s.3
s.4>>>>>>>>>>>>>;
```

```
  }
```

`Next` **transition to the next moment of time*

```
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT =(e.N)(e.K)
((e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT);
  }
```

`AxiMW` **axiom of moving the stepping switch after rotating the flow indicator*

```
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C
s.T)(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.R:'T',
<IncR (e.W)e.N>:e.WW = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R
s.P s.C s.T) ('I'e.XR)(e.XP)(e.WW) 'F' s.XP s.XC s.XT;
  e.1 = e.1;}
```

`AxiMT` **axiom of the pump doing back move*

```
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C
s.T)(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT,
s.P:'T'= (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)('I'e.XP)(e.XW) s.XR 'F' s.XC 'F';
  e.1 = e.1;}
```

`AxiMP` **axiom of the pump doing work move*

```
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C
s.T)(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.P:'F',
s.T:'T', <Neq (e.W)e.K>:'F'= (e.N)(e.K)((e.R)(e.P)(e.W)
s.R s.P s.C s.T)(e.XR)(e.XP)(e.XW) s.XR 'T' s.XC s.XT;
  e.1 = e.1;}
```

`AxiMR` **axiom of making the flow indicator move*

```
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C
s.T)(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.R:'F',
s.T:'T', <Neq (e.W)e.K>:'T'= (e.N)(e.K)((e.R)(e.P)(e.W)
s.R s.P s.C s.T) (e.XR)(e.XP (e.XW) 'T' s.XP s.XC s.XT;
```

```
  e.1 = e.1;}

AxiMC *axiom of unlocking the switch C
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C
s.T)(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'T',
s.T:'F', <Neq (e.W)'I'>:'T'= (e.N)(e.K)((e.R)(e.P)(e.W)
s.R s.P s.C s.T) (e.XR)(e.XP)(e.XW) s.XR s.XP 'F' s.XT;
  e.1 = e.1;}

AxiSC *axiom of moving the stepping switch due to C
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'F', <Neq
(e.W)e.N>:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W)s.R s.P s.C
s.T) (e.XR)(e.XP)('I') s.XR s.XP 'T' 'T';
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'F', <IncR
(e.W)e.N>:e.WW = (e.N)(e.K) ((e.R)(e.P)(e.W)s.R s.P s.C
s.T) (e.XR)(e.XP)(e.WW) s.XR s.XP 'T' s.XT;
  e.1 = e.1;}

AxiWStab *axiom of stability of the stepping switch
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'T',
s.R:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.W) s.XR s.XP s.XC s.XT;
  e.1 = e.1;}

AxiTStab *axiom of stability of the auxiliary switch T
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.T:'T',
s.P:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC 'T';
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.T:'F',
s.C:'T' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC 'F';
```

```
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.T:'F', <Neq
(e.W)e.N>:'T' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C
s.T) (e.XR)(e.XP)(e.XW) s.XR s.XP s.XC 'F';
  e.1 = e.1;}

AxiPStab *axiom of stability of the pump switch
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.P:'F', <Neq
(e.W)e.K>:'T' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C
s.T) (e.XR)(e.XP)(e.XW) s.XR s.P s.XC s.XT;
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.P:'F',
s.T:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.P s.XC s.XT;
  e.1 = e.1;}

AxiRStab *axiom of stability of the indicator switch
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.R:'F',
s.T:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.R s.XP s.XC s.XT;
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.R:'F', <Neq
(e.W)e.K>:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C
s.T) (e.XR)(e.XP)(e.XW) s.R s.XP s.XC s.XT;
  e.1 = e.1;}

AxiCStab *axiom of stability of the switch C
  {(e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'T', <Neq
(e.W)'I'>:'F' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C
s.T) (e.XR)(e.XP)(e.XW) s.XR s.XP s.C s.XT;
  (e.N)(e.K)((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.XC s.XT, s.C:'T',
s.T:'T' = (e.N)(e.K) ((e.R)(e.P)(e.W) s.R s.P s.C s.T)
(e.XR)(e.XP)(e.XW) s.XR s.XP s.C s.XT;
```

```
  e.1 = e.1;}

I *formatting a string to unary natural number
  {=;
  s.1 e.1 = 'I'<I e.1>;}

Neq *checking unequality of two natural numbers
  {() = 'F';
  (s.1 e.0)s.1 e.1 = <Neq (e.0)e.1>;
  (e.1)e.2 = 'T';}

IncR *increasing a natural number e.0 modulo e.1, plus 1
  {(e.0) e.1, <Neq (e.0)e.1>:'T' = 'I'e.0;
  (e.0) e.1 = 'I';}
```

Abstract Chaitin's theorem and its methodological consequences

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abstract. Abstract forms of Kolmogoroff's complexity, Chaitin and Gödel's theorems are stated. They are used to analyze numerous methodological issues: Kant's Third antinomy, Parkinson's law of committee, cooperative creative activity, multi-language programming, benevolence to other's views, dilemma of deism–atheism.

Keywords: Kolmogoroff complexity, Chaitin theorem, Gödel theorem, Kant antinomy, Parkinson law

Chaitin's theorem of unknowledgeable and Gödel's theorem of incompleteness are of great importance for philosophy of science. Numerous works around them are based on supposition that functions considered are algorithmic and objects are constructive (more specifically natural numbers). First of all we make some kind of reverse analysis to extend them into abstract domains and for wide class of theories. In essence we show that there are very weak assumptions on derivability of true formulas but somewhat stronger for underivability of false formulas.

1 Abstract computations

Definition 1. Let there is a finite set of letters $\mathbf{A}$. Lists of signature $\mathbf{A}$ are defined inductively:

1. Empty list `()` is denoted `NIL`.
2. Each letter is an atom.
3. If $a_1, \ldots, a_n$ are lists or atom then $(a_1 \ \ldots \ a_n)$ is a list.

A natural number n is represented by the list *(NIL ... NIL)* (n members). Thus 0 is *NIL*, 1 is `(NIL)` 2, is `(NIL NIL)` and so on.

DEFINITION 2. Functional signature is a finite set of atoms, including `cons`, `car`, `cdr`, `lh`, `members`, `id`, `concat`, `quote`, `arity`, `expand`, `join`, `perm`, `comp`, `const`.

Interpreter functional signature contains in addition to the above `turing`, `ifnil`, `ifatom`, `iflist`, `iffunction`, `equal`. Turing one contains also `eval`, full Turing one adds `search`. All these atoms are *elementary functions*. Lists in a functional signature are called *expressions*.

We interpret as a function the last member of a list.

Comment. It is for we consider Turing-incomplete languages where we cannot define a function to add an element to the tail of a list.

Some lists are functional ones. Some non-functional lists are convertible and can be computed.

DEFINITION 3. Let `F` is a functional list, `E` is any list. We can add integer indices which are written in the same string.

Elementary functions are functionals. Lists `(E F expand)`, `(E1 E2 F join)`, `(F1 E F2 comp)`, `(E const)`, `(E1 E2 turing)`, `(F1 F2 E ifnil)`, `(F1 F2 E ifatom)`, `(F1 F2 E iflist)`, `(F1 F2 E iffunction)`, `(E E1 eval)`, `(F F1 search)` are functionals.

Now a computational semantic of functions is defined. `arity` is applied to a functional and gives a number of arguments of a resulting function.
`(arity arity)` = 1.
`(E1 E2 cons)` computes both arguments and makes a list with the head E1 and the remaining part E2.
`(E car)` computes `E` and extracts its first element.
`(E cdr)` computes `E` and removes its first member.
`(E lh)` gives the number of symbols in the value of `E`. Each atom and each bracket is a single symbol.
`(E members)` gives the number of elements in the value of `E`.
`(E id)` gives a value of `E`.

`(E1 E2 concat)` joins two lists together.
`(E quote)` don't computes `E` and updates it as is.
`(F expand)` adds a fictive argument to the tail of arguments of `F`.
`(E1 E2 F join)` computes `E1, E2` and if their values are numbers in $1 \leqslant$ `Ei` $\leqslant$ `(F arity)` diminishes number of arguments of `F` by 1 glueing arguments with those numbers. `E1` is remaining in the list of arguments.
`(E1 E2 F perm)` is analogous permutating two arguments.
`(E F1 F2 comp)` substitutes `F1` for argument of `F2` with number `E`. $1 \leqslant$ `E` $\leqslant$ `(F2 arity)`. `((E F1 F2 comp) arity)` $=$ `(F1 arity)` $+$ `(F2 arity)` $- 1$.
`(E const)` is a function of arity 1 always giving value of `E`.
`(E1 E2 E3 turing)` computes `E2`, which is to be a functional, and then performs `E3` steps of its application to `E1` (`E3` is to have a number value) and gives a list `(E4 E5)`, where `E4` $= 0$, if computation had been finished on or before step `E3`, and 1 else. `E5` gives a result of (partial) computation. We accept that `(E1 0 E3 turing)`$=$`(1 E1)`.
`(E F1 F2 if[property])` computes `E` and if its result has a desired property gives `F1`, else `F2`. Arity of two functionals are to be equal.
`(E1 E2 equal)` gives 0, if results are literally the same, 1 else.
`(E1 E2 eval)` computes its arguments, second one is to be a functional, then applies this function to the value of the first argument.
`(F F1 search)` finds such tuple of values of arguments for `F`, for which `F` is equal to 0, and applies `F1` to a found values. Arities of functions are to be equal.

A list is *convertible*, if there is a subexpression which is not functional of the form `(E1 ... En F)`, where `(F arity)` $= n$.

There can be any number of extra elementary functions in our system. The only condition is that each function have a well defined computational semantics (not necessary algorithmic). Thus we defined a kernel language for different kinds of algorithmic and non-algorithmic computations (e. g. hyperarithmetic or computations on an algebraic structure).

PROPOSITION 1. *(limited λ-abstraction) Let us enrich our language by variables `x1, ..., xn`. Then for any list `E[x1,...,xn]` can be constructed a functional `FT` s.t. `(E1 ... En FT)`$=$`E[E1,...,En]`.*

PROOF is purely technical. □

PROPOSITION 2. *There holds a fixed point theorem in each Turing system: for each functional* `F` *there is such* `E`*, that for all* `E1`

$$\texttt{(E1 E F)} = \texttt{(E1 E eval)}.$$

PROOF.

```
LUR = (1 (eval const) (2 (NIL const) cons comp) comp)
  XXU = ( 1 2 (1 (1 LUR cons comp) cons comp) join);
            LXF = (1 (XXU const) F comp)
               YF = (1 LXF XXU comp).
```

`YF` is a fixed point.

□

PROPOSITION 3. *(Turing completeness) Turing systems allow to express any partial recursive function.*

PROOF. By fixed point and conditionals we can construct McCarthy recursive schemes.

Note now that `eval` is definable through `turing` and `search`. `eval` is called a universal function, `turing` is an interpreter, `search` is a search operator. No other dependencies hold for these three operators. Primitive recursive functions have an interpreter without search and universal function. Recursive schemata on real numbers and their lists with a signature $\{0.0, 1.0, =, >, +, *\}$ have universal function and interpreter but no search. Hyperarithmetical functions on real numbers have no search and no interpreter, only a universal function. Adding search we get no interpreter. Adding search to initial elementary functions gives no interpreter and no universal function. □

2 Generalization of Kolmogoroff complexity

DEFINITION 4. *Complexity* of an object relatively to a computational system is a minimal length of an expression which evaluates to our object. If a system is turing one, complexity is called kolmogoroff one. Complexity of an object x in a system Σ is denoted $(x\ \mathbb{K}_\Sigma)$.

If system is defined by a context it is omitted.

Definition 5. Let there are two computational systems $\Sigma 1$ and $\Sigma 2$. Coding `CODE[a]` of language of one system inside language of other is *regular*, if

$$(\texttt{CODE[a] lh}) \leqslant k \cdot (\texttt{a lh}) + C_1$$

C_1 — is a constant, k is a *coding factor*.
$\Sigma 1$ is interpreted in $\Sigma 2$, if there is a regular coding and a function `Int` such that

$$\exists n\, (\texttt{CODE[E] Int turing}_2) = (\texttt{0 CODE[a]}) \iff \texttt{E} = \texttt{a}.$$

$\Sigma 1$ is translated into $\Sigma 2$, if there is a regular coding and a function `Trans` such that

$$(\texttt{CODE[E] Trans eval}_2) = \texttt{CODE[a]} \iff \texttt{E} = \texttt{a}.$$

Theorem 1 (Kolmogoroff's theorem). *If $\Sigma 1$ is interpreted or translated into $\Sigma 2$ and k is a coding factor, then $k \cdot (\texttt{a}\ \mathbb{K}_1) \leqslant (\texttt{CODE[a]}\ \mathbb{K}_2) + C$.*

Proof is obvious. □

This theorem generalizes up to wide class of systems and codings (including Turing-incomplete and non-algorithmic) a theorem of Kolmogoroff on invariancy of complexity up to additive constant.

3 Generalization of Chaitin theorem

Let there is a theory **Th**, with definable predicates 'To be a natural number', $=$ and $<$ for natural numbers, constants $0, 1$ and functions $+, *, \uparrow$, the last one is a power function. Elementary arithmetical formulas are relations of two expressions in this vocabulary. Then we say that this theory contains natural numbers.

Let there is a full turing system Σ with functionals to test whether this list is a proof of a given formula in some regular coding, to extract a proved theorem from a proof code and to substitute an object of Σ (not necessarily a number) for a free variable of a formula and to compare two formulas textually.

DEFINITION 6. A theory is chaitin-correct w.r.t. Σ, if are expressible: a notion `(E E1 eval)=a`, a function `(a lh)`, all true formulas `(a lh)` $= n$ are provable, all closed true elementary arithmetical formulas are provable, and any closed false formula of the form $\neg$`(E E1 eval)=a` is not provable.

Each chaitin-correct theory is consistent. A simplest such theory $\mathbf{Ar}_0$ is given by the following axioms:

$$\begin{array}{ll} \forall x\,(x+0=x) & \forall x,y\,(x+(y+1)=(x+y)+1 \\ \forall x\,(x*0=0) & \forall x,y\,(x*(y+1)=(x*y)+x \\ \forall x\,(x\uparrow 0=1) & \forall x,y\,(x\uparrow(y+1)=(x\uparrow y)*x. \end{array}$$

THEOREM 2. *There is a number* $\boldsymbol{C}$ *in any chaitin-correct theory such that* $(a\ \mathbb{K}) > \boldsymbol{C}$ *is not provable for any* `a`*.*

PROOF. A formula expressing `(E E1 eval)=a` is denoted $R(p,x,a)$. Then a statement `(a` $\mathbb{K}$`)` $> C$ can be formulated as follows:

$$\forall x \forall p\,(\texttt{((x p) lh)} < C+1 \supset \neg R(p,x,a)).$$

If `(a` $\mathbb{K}$`)` $< C+1$ holds, then this formula is not provable inside **Th**, because elsewhere it would be provable a false statement `((x0 p0) lh)` $< C+1$ & $\neg R(p0,x0,a))$ and thus a false formula $\neg R(p0,x0,a))$ for some `((x0 p0) lh)` $< C+1$. Let show this and by the way construct a Chaitin's constant.

Let a functional `K` finds for each C searches a proof of a formula (`a` $\mathbb{K}$) $> C$ by brute force and if such proof is found gives `a`. Let a length of code for this functional is k. Let the quantity of different atoms in our system is m. Then there is such C_0, that $m^{C_0} > k*C_0$. This C_0 can be taken as a Caitin's constant. Let $(a\ \mathbb{K}) > C_0$ were provable. Then `K` would find such $a0$. But really $(a0\ \mathbb{K}) \leqslant C_0$ and thus `((x0 p0) lh)` $< C_0+1$ & $\neg R(p0,x0,a0)$ is not provable for some $p0$, $x0$. But $\forall x\forall p\,($`((x p) lh)` $< (C_0+1) \supset \neg R(p,x,a0))$ implies `((x0 p0) lh)` $< (C_0+1) \supset \neg R(p0,x0,a0)$. `((x0 p0) lh)` $< (C_0+1)$ is provable by correctness, therefore is provable $\neg R(p0,x0,a0)$. Contradiction. $\square$

This form of Chaitin's theorem does not demand computability of a system complexity is defined w.r.t. It uses search function essentially. It can be applied also for systems with infinite basic data type but with finite base of explicitly given atoms. Then complexity of some objects can be infinite (e.g. π in a system for algebraic operations on real numbers).

4 A generalized Gödel incompleteness theorem

Now we consider and generalize the Gödel incompleteness theorem in the form of Rosser [2]. Let we give some auxiliary definitions.

Definition 7. Restricted quantifiers are formulas of the form

$$\forall x\,(\texttt{(x lh)} < \mathbf{n} \supset A(x)), \quad \exists x\,(\texttt{(x lh)} < \mathbf{n}\ \&\ A(x)).$$

A formula $P(x))$ is *limitedly correct* in the theory **Th**, if from provability of $\exists x\,(\texttt{(x lh)} < \mathbf{n}\ \&\ P(x)) \vee B$ follows provability of $P(\texttt{a})$ for some $\texttt{(a lh)} < \mathbf{n}$ or provability of B itself.

Definition 8. A theory is *Gödel-correct* if a predicate $<$ is expressible for natural numbers; all closed true formulas of the form $\texttt{(a lh)} < \mathbf{n}$ are provable; there is some coding for formulas; there is a formula expressing 'p is a proof of $A(a)$' $Proof(p, \mathrm{CODE}[A], a)$; there is a functional to compute code of negation of a formula by its code $\texttt{Neg}$; if $A(a)$ is provable, then $Proof(p, \mathrm{CODE}[A], a)$ is provable for some p; a weak Gödel rule

$$(1) \quad \frac{Proof(p, \mathrm{CODE}[A], a)}{A(a)}$$

is admissible and $Proof(p, \mathrm{CODE}[A], a)$ is limitedly correct for all A, a.

Theorem 3. *If a theory is Gödel-correct it is incomplete.*

Proof. Consider a formula

$$(2) \quad \begin{array}{c} \forall x\,((Proof(x, z, z) \supset \\ \exists y\,((y\ \texttt{lh}) < (x\ \texttt{lh})\&Proof(y, (z\ \texttt{Neg}), z))))\& \\ \exists x\,((Proof(x, (z\ \texttt{Neg}), z) \\ \&\neg\exists y\,((y\ \texttt{lh}) < (x\ \texttt{lh})\&Proof(y, z, z)))) \end{array}$$

Substitute in it its code R. Then if the formula

$$\begin{array}{c}\forall x\,((Proof(x,R,R)\supset \\ \exists y\,((y\ \mathtt{lh})<(x\ \mathtt{lh})\ \&\ Proof(y,(R\ \mathtt{Neg}),R))))\ \& \\ \exists x\,((Proof(x,(R\ \mathtt{Neg}),R)\ \& \\ \neg\exists y\,((y\ \mathtt{lh})<(x\ \mathtt{lh})\ \&\ Proof(y,R,R))))\end{array} \tag{3}$$

is provable, we take a_0 with provable $Proof(a_0, R, R)$. Due to limitedly correctness of $Proof$ and by the first conjunctive subformula there is such $(\mathtt{a}_1\ \mathtt{lh}) < (\mathtt{a}_0\ \mathtt{lh})$, that $Proof(a_1, (R\ \mathtt{Neg}), R)$ is provable. Then by a rule (1) is provable a negation of (3) and our theory is inconsistent and proves everything. So it is not Gödel-correct.

If a negation of (3)

$$\begin{array}{c}\exists x\,((Proof(x,R,R)\ \& \\ \neg\exists y\,((y\ \mathtt{lh})<(x\ \mathtt{lh})\ \&\ Proof(y,(R\ \mathtt{Neg}),R))))\vee \\ \forall x\,((Proof(x,(R\ \mathtt{Neg}),R)\supset \\ \exists y\,((y\ \mathtt{lh})<(x\ \mathtt{lh})\ \&\ Proof(y,R,R)))),\end{array} \tag{4}$$

is provable then there is such b_0 for which $Proof(b_0, (R\ \mathtt{Neg}), R)$ is provable. From first disjunctive part follows

$$\exists x\ ((x\ \mathtt{lh}) < b_0 + 1\&(Proof(x,R,R))\,.$$

Applying limitedly correctness we get provability whether (3), which is contradictory, or the second disjunctive part. Then we get a contradiction analogously to the first part of proof. □

5 Philosophical consequences

Kant's Third Antinomy (of Freedom) can be substantiated precisely if complexity of a human is lower than complexity of the Universe. Parkinson's law of committee (decision of committee is more moronic than decision proposed of its stupidest member) can be proved precisely. One of paradoxes arising while applying precise Computer Science to real Informatics can be solved. It is known that Kolmogoroff's complexity is invariant up to ADDITIVE constant L. Using Chaitin's limit we can prove that the fixed constant L can substantially decrease the actual possibilities of programmer. Interrelation of Chaitin and Orevkov theorems yields that high level person can make things which cannot be understood by plain thinkers but to implement his/her insights plain thinking is often necessary. Some peculiarities of Chaitin's limit if person's mind is not Turing complete are considered.

6 Algorithmic randomness and Kant's Third Antinomy

So any formalism has limits such that upper them it cannot state a complexity of an object and thus cannot correctly comprehend and understand it. So an argumentation with complexity upper than Chaitin's limit for a person is understood by completely chaotic and illogical. But this is not the worst case. If such person tries to comprehend the arguments by cutting out all which cannot be placed in his/her head he/she gets an illusion of understanding together with completely wrong image of percept.

Chaitin [3] noted out that now existence of unknowledgeable is well substantiated and even proved. Each position based on supposition that human mind is omnipotent in principle is not even an opinion now. Our generalization of Chaitin theorem shows how weak premisses are sufficient for Chaitin's limit is existent. We do not need here to claim that human is a finite system which had been used in earlier demonstrations. This together with an observed harmony of the world substantiated theism in very high degree [4]. At the same time this shows that it is impossible to prove or to refute existence of God.

For finer methodological consequences it is reasonable to accept finiteness of a human (as for example in [7]). Thus because complexity of the Universe is much higher than one of a human and of the humanity (even in supposition that joining humans join only knowledge but not their ignorance). But incognizable can sometimes be partially appreciated. It is known that objects with big Kolmogoroff complexity are comprehended as random.

Kolmogoroff studied algorithmic randomness for infinite sequences (complexity of initial segment of a sequence will be same as its length up to additive constant). We are to define randomness of a finite object from the point of view of Chaitin's limit and his considerations in [4, 5]. This is randomness relative to a concrete object or subject processing information.

> *An object is random for a processor if its complexity is larger than processor's Chaitin's limit.*

Now we'll prove a proposition equivalent to Kant Third Antinomy [8] and even more strong, expressing it in the language of current science.

> *Human cannot state whether our Universe is deterministic or there is a necessary randomness in it.*

Let the Universe be deterministic. Then a complexity of the algorithm initialized during world's creation is higher than Chaitin's limit of humanity. Thus humanity cannot comprehend a Word's idea as a whole and complete entity. Deterministic world is understood as random one.

Note!!! We are not creationists here. World creation would be a natural process for example as a garbage of a super-civilization during re-creation or transformation of its own World (S. Lem: From Einsteinian to Testan Universe. In Polish).

Let our World be indeterministic. If we were proved this we were proved that complexity of our World is higher than Chaitin's limit of our civilization. This is a contradiction.

Thus problem whether our Universe is deterministic is a pseudoproblem from the point of view of pure exact knowledge. We are free to choose a theory which in the moment is a best fit for 'practice' and is a better representation of objects in view.

Therefore it is inacceptable to advertise results of our science as 'scientific truth'. They are to be re-verified by an alternative theory. This is a strong opposition for postmodernistic 'tyranny of truth'. We cannot lay our responsibility on arms of Science or God.

7 Parkinson's law

Let there is a committee which is to work out a decision understandable for all its members for each could meaningfully vote 'yea' or 'nay'. In this case Chaitin's limits of committee members are to be reduced to minimal one because else some of members cannot understand a proposal. So a weak Parkinson's principle is substantiated:

Weak Parkinson's law:

Decision of a committee is no more adequate that one which could make the least competent of its members himself.

But the reality is more crude. Each committee member has different competentions in different domains. So we need to introduce a matrix of limits. If two limits of persons are C_i and C_j, complexities of translations from one system of notions into an other are K_{ij} and K_{ji}, then maximal complexity of a decision of each of them understandable by both is $C_{ij} = \min\{C_i - K_{ji}, C_j - K_{ij}\}$: a limit of i-th person for understanding of j-th. Thus even not taking into account non-uniformity of knowledge inside a Chaitin limit we get the following upper bound: $\min_{i,j} C_{ij}$. We substantiated the following

Strong Parkinson's law:

Decision of a committee is more moronic that a decision which could make the most moronic of its members himself.

In Venice and Rome important decisions were delegated to a truthful person which had been made fully responsible for its realization and consequences…

8 Chaitin limit and paradox of inventor (Orevkov theorem)

There is at least one more quality of mind orthogonal to brute force which can lead to relatively large Chaitin's limit. This is ability to master complex notions.

Orevkov theorem (1968): Indirect proof in logic can be in the tower of exponents times shorter than direct one.

Orevkov's theorem is a precise partial case of a general paradox of inventor formulated by Gy. Polya:

To prove a simple statement we are often to use complex intermediate notions. To prove a weaker and 'simpler' statement can be much more harder than to prove more strong and complex one.

Gy. Polya pointed out and partially explained this paradox w.r.t. inductive proofs. Orevkov substantiated that it is a fundamental property of thinking.

Using high order notions we can jump far away behind Chaitin's limit of crawling persons. This substantiates a genial insight of D. Hilbert that ideal notions are necessary to obtain non-trivial practical (real) results.

American scientist M. Furman wrote (private communication discussing my preliminary notes on Chaitin's limit): 'Non-equivalence (not considering purely theoretical notion of Kolmogorov complexity, but from the point of view of real application) is defined by resources: size of memory and execution time.

Theoretically we have two binary properties: is memory finite or is time finite. But seeing one step deeper we understand that there is a uniform restriction for some class of examples'.

These arguments do not disturb our basic considerations and only show that real situation is even more fine and interesting. It is known that primary resource of human defines his/her logic (linear logic is logic of money$<$ intuitionistic one is logic of knowledge, nilpotent one is logic of time and so on). Of course it can restrict Chaitin's horizon even more substantially than Kolmogoroff complexity.

M. Furman also proposed an example showing interconnections of Chaitin's limit with inventor's paradox. If a person mastered a high-level method he can say something like to Furman's objection: 'It is very easy to construct a translator having the precise definition of a language'. But method of formal semantics itself cannot be treated as a simple one. And it is known how hard is to write out a formal definition of a semantic.

Evgeny Kochurov pointed out (private communication) that usually those who cannot comprehend complex notions but have a big operative memory can build long and relatively complex first-order compositions. Those who excellently appreciate methods can find excellent critical points but poorly analyses a crawling process how to go from one critical point to next one. So those two are complementary and can excellently assist one another if each person is used according to his/her strong sides. So we transferred to a problem how to avoid Parkinson's law.

9 Consequences for organisation of creative work: How to avoid Parkinson's law?

There is an interesting example which seems to be a strong counterexample to Parkinson's law. Each bee, termite or ant acts like finite automaton with a fixed program and low memory. Nevertheless a general behaviour of nest become very complex and adaptive. Moreover ants for example demonstrate more complex forms of integration and system behaviour. Remember ant empires joining in the single net thousands of nests which have intensive exchange of information, people and genetic material (trade points and exchange of nymphs).

We apply here an analogy from logic. Von Neumann's theory of self-reproducing automata shows how to compose an upcoming system from uniform units with extremely simple behaviour. Thus a good organization of morons which cannot understand even loops can generate recursions and high level constructions.

How is it possible? It is because cooperation itself is performed by strict simple automata rules. This analogy is used in neuron nets in such domains as pattern recognition in cases when there are no precise algorithms. Well trained neuron net mistakes sometimes but rarely. And nobody knows why.

Ideology of crowdsourcing tries to transfer this experience into human society. But as for neuron nets here we get no creativity[1]. How to introduce it?

As usually direct and obvious decision — to make automata stochastic or indeterministic — fails here. Such approach to creation process is fantastically ineffective.

So we come to a tough consequence for human collectivities. Committee consisting from equal and free creative persons is im-

[1] In Russian there are two words for English 'creativity'. Креативность (creativity) means invention of something new only to be new without real values and goals. Творчество means creation of new and useful things. This is why 'creative class' is appreciated by Russians as class of uppity, spiritually and really impotent egocentric persons.

potent. Potent can be at least two-level structure. Interactions are strictly formalized on first level and for connections between first and second level. In contrary interactions on second level are bounded by clear and ruthless rituals but never formalized. They are diminished to a reasonable minimum. Upper level is responsible for creative decisions and lower for their realization. It is often possible to implement an idea inside a rigid structure but never is possible to get a new idea here.

We have here another 'counterexample': freesofters. This seems to be a conglomerate of free creative individuals which interact very informally. But this is not the case. They curse and laud one another very informally but their interactions in coding, bug processing, documentation and so on follow strict rules. So I cannot say that they are 'free persons' in vulgar sense of this word. They are free individuals having real goals and values and voluntarily sacrificing some 'freedoms' for those high valuables. They can be an embryo of a structure which can save humanity and some real achievements of current ill civilization after its inevitable death.

And now in a cold water. A community of freesofters can be so effective because almost all of them are involved into really non-creative problems of coding according to existing algorithms and architects, debugging and developing earlier projects. But this community has also an ecological niche for really creative persons.

Warning. A society based on freesofters-like libertarian principles will ruthlessly apply 'measures of humanitarian defence' (see e.g. A. A. Rosoff 'Confederation Meganesia') and suppress minorities which wish to claim their rights in manner restricting other people's rights and common values. It may be necessary to survive against mindless hordes of 'free vultures'.

Furthermore collective intellect of best algebraists allowed to solve a problem of classification of finite groups [9]. But interaction of professional pure mathematicians is so deeply ritualized[2] that this example is a verifying example for us.

These examples allow us to make principle of committee more precise. Committee must elaborate a decision. Such decision will inevitably be a compromise e.g. a mixture of unpleasant and useless.

[2]And not formalized, in contrary to common prejudice.

Creative persons try to find a solution. They do not try to cut it according to lower level of their understanding. In contrary, people develop an other's people nice idea even they do not appreciate it as a whole and often find new aspects of it. So a good organized creative storming can lead to a valuable results. High level people know how useful is a discussion of equal in spirit and mind persons (but not those nominated by an institution).

Collective creative work is development and transformation of new ideas without 'full comprehension'.

How to increase effectivity of this storming?

1. Sacrifice sacred cows.
2. Make hidden conceptual contradictions visible.
3. Don't pronounce 'universal and indisputable truths' (BLAGOGLUPOSTI (in Russian) I don't know an analogy in English).

All these three points contradict to politcorrectness and other liberal taboos.

10 Chaitin limit and programming languages

Formally complexity of programs in different PL is equivalent up to additive constant (Kolmogoroff theorem). Practice shows the opposite: program written by adequate tools can be 50 times shorter than in 'universal' Java or C# Why?

Kolmogoroff's theorem (1) states that $k\cdot(\mathtt{a}\ \mathbb{K}_1) \leqslant (\mathtt{CODE[a]}\ \mathbb{K}_2)+C$ where k is equal to 1 if we consider standard programming codes. Constant C is a length of a translator program for the second language written in the first language. To write it eats almost all Chaitin's limit of a programmer.

Therefore we have an excellent and precise demagogic answer on a moronic and demagogic question very often posed to ones who did something by 'exotic' language: 'Is it possible to write the same in C# or Java?':

— Of course. It is possible to write all in the language of Turing machines, if you prefer.

Thus theoretical equivalence sometimes means practical incomparability.

This analogy works in other domains also. If we do not master a language of a concrete domain we can *in principle* to understand constructions and arguments but it is necessary to build in our mind a 'translator' into our paradigm. Its complexity can be so high that it leaves almost no resources to analyze the argumentation.

Another warning. If you know many languages but have no background fundamental knowledge in your head you work worse that blind coder. Multi-tool method is effective only when a person masters a meta-knowledge, meta-method and a basis of notions.

So fundamental knowledge is that which forms a system in a brain. Foundation of a system must be stable. It consists of a basis of relatively simple notions (keystones) amalgamated by a lot of relation and properties which show their interrelations gains, shortcomings and restrictions. It is ideal if in result a person sees restrictions of his/her system as a whole.

And there is one more bad side. Many people simply cannot appreciate complex (algorithmic) constructions such as recursions and even loops. They have no universal algorithm in their head. Here Chaitin's limit is 0 and this person simply can see nothing.

Final remark

It is false that clever one works faster than more stupid one. A stupid person never can understand what does a clever one and never can make the same work.

11 Benevolence to other's views

A problem of co-existence of different views is madly contaminated by 'tolerance' originated in the fundamental mistake of J. S. Mill: he declared freedom of opinions instead of freedom of argumentation. He simply could not imagine that every irresponsible and moronic cry will demand rights and honors because it is an 'opinion of a free person'.

This goes deeper to BLAGOGLUPOST of Voltaire's 'I hate your opinions, but I would die to defend your right to express them'. We see that there are too much people who accept no counter-arguments against their opinions but are ready to kill each who criticizes them. We see that there are too much people and institutions which substantiate their opinions not by argumentation but by direct lie and manipulations (e.g. neo-liberals, neo-cons, fundamentalists, juvenile justice …).

Principle of benevolence to other's views.

Remember that The Truth is inaccessible to you and to any other human. Thus say confronting other views.

I do not agree with your views but you argue in their favor honestly and earnestly. Thus I will defend your right to proclaim them, to substantiate them and to distribute them. In the same time I declare full and unrestricted right of me and any other person to criticize them, to find weak points in your argumentations and maybe lie and manipulations.

This obligation is ended when your sights become refuted or you are catched on lie or manipulations (sophistic or psychologic).

In the first case you remain an honest person for me and I will defend you against any attempts to punish you for error itself (but not for its consequences). If you will be so brave to recognize you have been mistaken I will help you to correct it and its consequences and you will become greater in my eyes.

If you will be catched on dishonored tricks all my responsibility will end. I will support the toughest of possible legal punishments for you because spiritual poison is more mortal than material.

12 Methodological argument for deism

Chaitin's theorem showed that Kant was right stating that our intellect cannot solve a problem of God's existence. So we have the following consequences.

1. Existence of God is a pseudoproblem from scientific point of view and you must take your own decision here.

2. It is unacceptable to cry that science rejects God (and equally that science proves God's existence).

3. It is inadmissible to make any scientific consequences from existence or non-existence of God.

4. It is acceptable to analyze this problem methodologically.

So the problem of deism or atheism is a methodological problem. Stating a rational definition of God as *The Truth*, as *the unified highest law of both nature and spirit which is beyond all worlds and all times* we are inspired to find unity in difference, high level unifying notions and principles for realizations which seem to be not connected for plain thinking, or even contradictory though both existing. It inspires us to develop ourselves both intellectually and spiritually and to keep these different sides and our material being in harmony.

In contrary atheism demotivates us to idolize and adore our imperfect plain reasoning and our restricted knowledge.

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Successful science without miracles

Ilkka Niiniluoto

abstract. Science is highly successful in making empirical predictions and guiding our practical actions. This paper defends the so-called 'ultimate argument for scientific realism' by claiming that this empirical and pragmatic success of scientific theories would be a miracle unless they are true or truthlike. This argument is abductive in Charles Peirce's sense, as it appeals to inference to the best explanation.

Keywords: abduction, explanation, fallibilism, inference to the best explanation, scientific realism, truthlikeness

It is generally agreed that science is highly successful in making empirical predictions and guiding our practical actions. The so-called 'ultimate argument for scientific realism' claims that this empirical and pragmatic success of scientific theories would be a miracle unless they are true. This argument is abductive in Peirce's sense, as it appeals to inference to the best explanation. This paper considers the idea that abductive inference can be reformulated by taking its conclusion to concern the truthlikeness of a hypothetical theory on the basis of its success in explanation and prediction. The strength of such a fallible argument is measured by the estimated verisimilitude of its conclusion given the premises.

1 Critical Scientific Realism

Scientific realism as a philosophical view has (i) ontological, (ii) semantical, (iii) epistemological, (iv) theoretical, and (v) methodological aspects (see [16], [29]). It holds that (i) at least part of reality is ontologically independent of human mind and culture. It takes (ii) truth to involve a non-epistemic relation between language and reality. It claims that (iii) knowledge about mind-independent (as well as mind-dependent) reality is possible, and that (iv) the best and deepest part of such knowledge is provided by empirically testable scientific theories. An important aim of science is (v) to find true and informative theories which postulate non-observable entities to explain observable phenomena.

Critical scientific realism can be distinguished from metaphysical or naive forms of realism by the principle of *fallibilism*: all factual human knowledge is uncertain or corrigible. Even the best results of science may be false, but still they may be probable, truthlike or approximately true.

Critical scientific realists have argued — following Charles S. Peirce [27], *pace* opponents like W.V.O. Quine and Larry Laudan (see [10]) — that it indeed makes sense to say that one hypothetical (even false) theory is 'closer to the truth' than another theory. By the same token, it is meaningful to state that a sequence of theories 'approaches to the truth', even when the final limit is not reached. Since 1974, after Karl Popper's 1960 attempt to define verisimilitude turned out to fail, the notion of similarity between states of affairs has been employed to give a precise definition of truthlikeness for scientific statements (see [15], [8]). The *degree of truthlikeness* Tr(H, C*) of a theory H is defined relative to a chosen target C*, where C* is the complete truth expressible in a given conceptual framework. Tr(H,C*) has its maximum value 1 when H is identical with C*. Such objective but usually unknown degrees of truthlikeness can be estimated by the *expected degree of truthlikeness* ver(H/E) of H given available evidence E (see [15, p. 269]). This measure is an epistemic indicator of objective truthlikeness in the same sense as posterior probability P(H/E) of H given E is an empirical indicator of the truth of H. For a logical truth H, we have P(H/E) = 1 but ver(H/E) $<$ 1, since H is not informative. On the other hand, ver(H/E) may be non-zero, and even high, when P(H/E) = 0. Thus, while ver involves epistemic probabilities, it is not identical with posterior probability. Given ideal conditions about the correctness and completeness of evidence E, it can be shown that ver(H/E) approaches the real degree of truthlikeness Tr(H,C*) of H (see [21], [25]).

The notion of truthlikeness does not replace the objective concept of truth, but rather presupposes the correspondence theory of truth as explicated in Tarski's model-theoretic definition (see [16]). By combining the goals of truth and information, it helps the scientific realist to define *scientific progress* as theory-change with increasing truthlikeness (see [14]).

Laudan's ([10]) 'pessimistic meta-induction' is based on the premise that many theories in the history of science have been non-referring and false but yet to some extent empirically successful. By induction, one might infer that this is the fate of our current and future theories as well. However, instead of simply concluding that future theories are false, the realist can argue that in typical cases the successor theory is more truthlike than its predecessor. For example, even though many scientific theories contain idealizations, which are known to be false, the powerful method of 'concretization' helps to remove such assumptions and thereby lead us toward the truth (see [8], [20]). This comparative and dynamic picture of progressive science evades the pessimistic conclusion that all present and future theories are far from the truth.

2 Abduction and the No Miracle Argument

In his fallibilist analysis of inference, Peirce argued that science uses, besides deduction, also two ampliative forms of reasoning: induction and abduction. *Abduction* is reasoning from effects to causes, or from observational data to hypothetical explanatory theories:

(1) The surprising fact E is observed;
But if H were true, E would be a matter of course.
Hence, there is reason to suspect that H is true.

[27, 5.189]. Against Comte's positivism, Peirce claimed that abduction frequently supposes 'something which it would be impossible for us to observe directly' [27, 2.640].

Peirce insisted that abduction or 'inference to an explanation' has a significant role in science. Often this role has been interpreted as the heuristic function of the *discovery* of new theories (N. R. Hanson), or alternatively as the motive for suggesting or *pursuing* testworthy hypotheses. Peirce further pointed out that in science the abductive step is followed by severe observational and empirical tests of the deductive or probable consequences the hypothesis [27, 2.634]. The examples of abduction range from compelling everyday observations to the adoption of theoretical hypotheses in science by virtue of their explanatory and predictive power. In these cases, it appears that sometimes abductive arguments can serve in providing a fallible *justification* of a hypothesis. Along these lines, Peirce's schema (1) has been interpreted by Gilbert Harman as *inference to the best explanation* (IBE).

For a critical realist, it is interesting to study the idea that abductive inference (1) can be reformulated by taking its conclusion to concern the *truthlikeness* of a hypothetical theory on the basis of its success in explanation and prediction (see [7], [18], [19]). This modification of abduction is also relevant to what Alan Musgrave [13] calls the 'ultimate argument for scientific realism'. After the 1950s, when scientific realism became a tenable position after the dominance of empiricism and instrumentalism, several philosophers of science (among them Jack Smart, Hilary Putnam, Grower Maxwell, and Richard Boyd) have defended realism as the best hypothesis which explains the practical (empirical and pragmatic) success of science. The ability of scientific theories to explain surprising phenomena and to yield correct empirical predictions and effective rules of action would be a 'cosmic coincidence' or a 'miracle' unless they refer to real things and are true or at least approximately true or truthlike (see [29], [30]). It is clear that the form of this 'no miracle argument for scientific realism' (NMA) is abductive (see [14, p. 51]).

In his well-known 'confutation of scientific realism', Laudan [10] demanded the realists to show that there is an 'upward path', or an epistemic warrant, from the empirical success of science to the approximate truth of theories — and then a 'downward path' from approximate truth to empirical success. In this paper, I restrict my remarks to the upward path (cf. [14, Ch.7]). For the downward explanation of the empirical success of science by the truth or truthlikeness of theories, and for arguments against alternative putative explanations (cf. [6], [33]), see [16, pp. 192–198]. Theo Kuipers [8] also gives a reply to Laudan by his 'downward' Success Theorem and 'upward' Rule of Success.

Both Laudan's challenge and the no miracle argument as a reply to this challenge presuppose a minimal realist framework where it makes sense to assign truth values to scientific statements (including theoretical postulates and laws). Besides semantic realists, this framework is accepted by such methodological and epistemological anti-realists who think that the truth of theories is an irrelevant [33] or 'utopian' aim [10] which 'exceeds our grasp' [32]. If successful, the no miracle argument is also relevant to those semantic anti-realists and instrumentalists whose inclination to treat theories as schemata without truth values is motivated by their belief about the inaccessibility of theoretical truth.

3 The Justification of Abduction

The idea about the justification of abduction has been understood in three different senses.The first is Peirce's own account of *truth-frequency*, later followed by many frequentist theories of probability and statistics in the 20th century (cf. [3]). The second approach is the qualitative theory of *confirmation* (cf. [31]). The third approach is the Bayesian theory of inference in terms of epistemic probabilities (see [17]).

Assume that an epistemic probability measure P is available for the scientific language, and define confirmation by the *Positive Relevance* criterion: E confirms H if and only if $P(H/E) > P(H)$. Then, by Bayes's Theorem,

(2) If H logically entails E, and if $P(H) > 0$ and $P(E) < 1$, then $P(H/E) > P(H)$.

This result is the basic principle of the *hypothetico-deductive* (HD) *method* in science. More generally, as positive relevance is a symmetric relation, it is sufficient for the confirmation of H by E that H is positively relevant to E. If inductive explanation is defined by the positive relevance condition, i.e., by requiring that $P(E/H) > P(E)$ (see [26], [5]), then we have the general result:

(3) If H deductively or inductively explains E, then E confirms H.

The same principle holds for empirical predictions as well, so that (2) can be generalized to the *hypothetico-inductive* (HI) or *hypothetico-probabilistic* (HP) method ([26], [9]). Hence, by (2) and (3), *empirical success confirms the truth of a hypothesis.*

It is important that in (3) H may be a theory expressed in theoretical terms beyond the observational language. If degrees of confirmation are measured by the difference between posterior and prior probability, i.e., P(H/E) − P(H), then evidence E gives strongest support to the minimal explanation H that is needed to account for E without irrelevant additions (see [19],[21]). Theoretical postulates are typically needed for such a minimal explanation, as theoretical terms can be logically indispensable for inductive systematization of observation statements (see [26]).

The notion of confirmation is still weak in the sense that the same evidence may confirm many alternative rival hypotheses. It is clear that for given evidence E one can always conceive many false premises from which E is derivable. A good theoretical explanation should be initially *plausible* relative other accepted theories, and it should not only account 'locally' for the given E, but it also should be *independently testable* by new kind of evidence. Indeed, it can be shown that the confirmation of a theory H increases if it is able to explain in a unified way many independent phenomena (see [23]). But a confirmed hypothesis need not yet be rationally and tentatively *acceptable* on evidence. A stronger notion of inference is obtained if one of the rival hypotheses is the *best* explanation of the facts. The strongest justification is obtained if the hypothesis is the *only* available explanation of the known facts. The Bayesian approach immediately shows that P(H/E) may be close to 1 and P(∼H/E) close to 0, when H is the only explanation of E. This suggests that abduction, or Inference to the Best Explanation, might be formulated as a rule of acceptance:

(IBE) A hypothesis H may be inferred from evidence E when H is a better explanation of E than any other rival hypothesis.

Comparison with Peirce's schema (1) suggests the following version of IBE:

(IBE′) If hypothesis H is the best explanation of evidence E, then conclude for the time being that H is true.

In analyzing IBE′, it is useful to follow Peirce in distinguishing between *deductive* and *inductive-probabilistic* explanations (cf. (3)) (see [17]). But one should also allow *approximate explanations*: H approximately explains E when it is possible to derive from hypothesis H something E′ which is close to E. Indeed, the empirical success of scientific theories in explanation and prediction is often approximate in this sense. For example, Newton's theory explains approximately the laws of Kepler and Galileo. However, here the evidence may still indicate that the best hypothesis is *truthlike*. This principle might be called *inference to the best approximate explanation*:

(IBAE) If the best available explanation H of evidence E is approximate, conclude for the time being that H is truthlike.

If degrees of truthlikeness are introduced, then there is a natural addition to IBAE: the greater the fit between H and E, the larger the degree of truthlikeness of H in the conclusion. (This gives an answer to P. Kyle Stanford's criticism of Jarrett Leplin's account of partial truth in [32, p. 158].)

By combining the ideas in IBE′ and IBAE, *inference to the best theory* can be formulated by the rule

(IBT) If theory H is the best explanation of evidence E, conclude for the time being that H is truthlike.

Here the acceptance of H is understood in the fallibilist sense that H is taken to be an informative theory close to the truth. In a comparative formulation,

(IBT^c) If H′ is a better explanation of evidence E than H, conclude that H′ is more truthlike than H.

(See also [7, 8].)

Many attempts to defend scientific realism by the no miracle argument NMA appeal to forms of abduction which conclude that successful scientific theories are approximately true, without making the notion of approximate truth precise (e.g., Putnam, Psillos). In a general form this argument looks like the following:

(NMA) Many theories in science are empirically successful.
The truth or truthlikeness of scientific theories is the best explanation of their empirical success.
Hence, conclude that such successful theories are truthlike.

The same argument can be applied to particular scientific theories. The first premise about the success of science is accepted both by realist and anti-realists, even though in particular cases the attribution of success to a specific theory may be non-trivial (e.g., it may be a matter of controversy whether a medicine, treatment or therapy is really causally effective in producing the desired results). As a whole, the argument NMA involves something like the principle IBT, and the conclusion supports the position of critical scientific realism.

A comparative version of NMA can be given as follows:

(NMA^c) Theory H' is empirically more successful than its rival H.
That H' is more successful than H can be explained by the assumption that H' is more truthlike than H.
Hence, conclude that H' is more truthlike than H.

To save the no miracle argument NMA against the charges of circularity ([10], [6]) and incoherence ([33]), one needs to defend abduction in the form of IBT or IBT^c.

4 Upward Inference and Expected Truthlikeness

The probabilistic account of IBE, given by the results (2) and (3), establishes *a probabilistic link between explanatory power and truth*: posterior probability P(H/E) is the rational degree of belief in the truth of H on the basis of E, and thereby confirmation, i.e., increase of probability by new evidence, means that we rationally become more certain of the truth of H than before. But a rule of the form IBAE needs a link between approximate explanation and truthlikeness. The notion of probability (at least alone) does not help us, since the approximate explanation of E by H allows that H is inconsistent with E, so that P(E/H) and P(H/E) are zero. Also for the treatment of IBT, we need a method for assessing the truthlikeness of a theory given empirical evidence. Here the notion of expected truthlikeness ver(H/E) can be used as an *empirical indicator of truthlikeness.*

Expected verisimilitude helps to define a notion of ver-confirmation in analogy with positive relevance: ver(H/E) $>$ ver(H) ([5], [21]). Then we have, for example, the following result:

(4) If H entails E but ∼H does not entail E, then E ver-confirms H.

This conclusion is still weak. It does not exclude the possibility that the 'catch-all' hypothesis ∼H includes 'unconceived alternatives' to H which also explain E (see [32]; cf. [30]). However, in cases of 'underdetermination' between rival explanations H and H′, which seem to account for the available evidence E equally well, the scientific strategy is to expand the evidence E with new observations, instruments, and active experiments, so that eventually a difference in the empirical success of H and H′ is revealed. A powerful mathematical theorem, proved by Johann Radon already in 1917 and today applied in various kinds of abductive 'inverse problems', shows under what conditions evidence guarantees the existence of a unique 'backward solution' (see [24]).

Another application of ver is to use expected verisimilitude as a criterion of acceptance. This is in harmony with the suggestion that the strength of IBT is assessed in terms of the expected verisimilitude of its conclusion given the premises. Thus, in order to reply to Laudan's 'upward' challenge, we should investigate whether the following kinds of principles are valid:

(5) If H′ is a better approximate explanation of E than H, then ver(H′/E) $>$ ver(H/E).

(6) If H approximately explains E, then ver(H/E) is high.

(7) If H is the best available explanation of E, then ver(H/E) is high.

(Cf. [22].) These results, which can be proved at least in special cases (see [19]), show that explanatory success gives us a rational warrant for making claims about truthlikeness. Thereby the notion of expected truthlikeness, explicated by the function ver, provides *a fallible link from the empirical success of a theory to its truthlikeness.*

Under ideal conditions, where a high value of ver(H/E) guarantees that the objective degree of truthlikeness Tr(H,C*) is also high, results (5)–(7) show that the method of accepting theories with maximal estimated verisimilitude is 'functional for truth approximation' in the sense of Kuipers [8, 9] (cf. [25]).

It is important to emphasize the fallible nature of results like (2), (3), and (5)–(7). The notions of confirmation and expected verisimilitude are historical, relative to the rival theories and evidence available at a given time. Some philosophers have continued Laudan's pessimistic argument, in many cases against formulations of Stathis Psillos [29], by giving historical examples of past theories which had some empirical success, including novel successes in relation to their predecessors, but still are non-referring and false by present lights. Against the claims of 'preservative' or 'localized' realism, such successes may have been based upon theoretical postulates that are discredited today (see [2], [4], [12], [32]). However, critical realists may acknowledge that, for example, relative to the historical situation the caloric theory of heat was well supported by the available evidence. By NMA^c, such theories were progressive in relation to their predecessors. (For the case of phlogiston theory, see [16, pp. 191–192]; for old quantum theory, see [8, pp. 278–288].) The fact that such theories have been replaced by better theories is not a 'Pyrrhic victory' for scientific realism (see [32]), since it supports the realist picture of scientific progress as increasing truthlikeness.

In fact, the measure of expected verisimilitude can be used also for retrospective comparisons, if the evidence E is taken to include our currently accepted theory T, i.e., the truthlikeness of a past theory H is estimated by ver(H/E&T) (see [14, p. 171]). In a similar way, Jeffrey Barrett [1] has proposed that — assuming that science makes progress toward the truth through the elimination of descriptive error — the 'probable approximate truth' of Newtonian gravitation can be warranted by its 'nesting relations' to the general theory of relativity.

5 Conclusion

Non-scientific explanations of the success of science — e.g. appeal to miracles or God's will — are not satisfactory. Therefore, we may conclude that scientific realism is the *only* explanation of the empirical success of science. This strong form of IBE justifies the no miracle argument NMA, and thereby gives us the best defence of scientific realism.

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Between $\mathrm{Int}_{<\omega,\omega>}$ and intuitionistic propositional logic[1]

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ABSTRACT. This short paper presents a new domain of logical investigations.

Keywords: paralogic, paracomplete logic, paraconsistent logic, paranormal logic, intuitionistic propositional logic

The language L of each logic in the paper is a standard propositional language whose alphabet is as follows: $\{\&, \vee, \supset, \neg, (,), p_1, p_2, p_3, \dots\}$. As it is expected, $\&$, $\vee$, $\supset$ are binary logical connectives in L, $\neg$ is a unary logical connective in L, brackets (,) are technical symbols in L and $p_1, p_2, p_3, \dots$ are propositional variables in L. A definition of L-formula is as usual. Below, we say 'formula' instead of 'L-formula' only and adopt the convention on omitting brackets. A formula is said to be quasi-elemental iff no logical connective in L other than $\neg$ occurs in it. A length of a formula A is, traditionally, said to be the number of all occurrences of the logical connectives in L in A. A logic is said to be a non-empty set of formulas closed under the rule of modus ponens in L and the rule of substitution of a formula into a formula instead of a propositional variable in L.

Let us agree that α and β are arbitrary elements in $\{0, 1, 2, 3, \dots \omega\}$. We define calculus $\mathrm{HInt}_{<\alpha,\beta>}$. This calculus is a Hilbert-type calculus, the language of $\mathrm{HInt}_{<\alpha,\beta>}$ is L. $\mathrm{HInt}_{<\alpha,\beta>}$ has the rule of modus ponens in L as the only rule of inference. The notion of a proof in $\mathrm{HInt}_{<\alpha,\beta>}$ and the notion of a formula provable

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in this calculus are defined as usual. Now we only need to define the set of axioms of $\mathrm{HInt}_{<\alpha,\beta>}$.

A formula belongs to the set of axioms of calculus $\mathrm{HInt}_{<\alpha,\beta>}$ iff it is one of the following forms (A, B, C denote formulas):

(I) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$, (II) $A \supset (A \vee B)$, (III) $B \supset (A \vee B)$, (IV) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$, (V) $(A \& B) \supset A$, (VI) $(A \& B) \supset B$, (VII) $(C \supset A) \supset ((C \supset B) \supset (C \supset (A \& B)))$, (VIII) $(A \supset (B \supset C)) \supset ((A \& B) \supset C)$, (IX) $((A \& B) \supset C) \supset (A \supset (B \supset C))$, (X, α) $\neg D \supset (D \supset A)$, where D is formula which is not a quasi-elemental formula of a length less than α, (XI, β) $(E \supset \neg(B \supset B)) \supset \neg E$, where E is formula which is not a quasi-elemental formula of a length less than β.

Let us agree that, for any j and k in $\{0, 1, 2, 3, \ldots \omega\}$, $\mathrm{Int}_{<j,k>}$ is the set of formulas provable in $\mathrm{HInt}_{<j,k>}$. It is clear that, for any j and k in $\{0, 1, 2, 3, \ldots \omega\}$, a set $\mathrm{Int}_{<j,k>}$ is a logic. It is proved that $\mathrm{Int}_{<0,0>}$ is the set of intuitionistic tautologies in L (that is, the intuitionistic propositional logic in L). By S we denote the set of all logics which include logic $\mathrm{Int}_{<\omega,\omega>}$ and are included in $\mathrm{Int}_{<0,0>}$ and by *ParaInt* we denote $S \setminus \{\mathrm{Int}_{<0,0>}\}$. Note logic $\mathrm{Int}_{<\omega,\omega>}$ is the intersection of all logics, other than itself, in *ParaInt*. The set *ParaInt* is of interest for scholars who study paralogics (paraconsistent or paracomplete logics). The set *ParaInt* contains (1) a continuous set of paraconsistent, but non-paracomplete logics, (2) a continuous set of paracomplete, but non-paraconsistent logics, (3) a continuous set of paranormal logics. We have some results concerning both logics from *ParaInt* and classes of such logics. In particular, we have methods to construct axiomatisations (sequent calculus and analytic-tableaux calculus) and semantics (in the sense of Kripke) for any logic $\mathrm{Int}_{<j,k>}$, where j and k in $\{0, 1, 2, 3, \ldots \omega\}$.

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Dynamic logic versus GTS: A case study[1]

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abstract. In this paper I will compare several solutions to a well known puzzle: Monty Hall. This will enable us to illustrate varitous styles of logical reasoning, and in particular to compare dynamic logic with game-theoretical approaches.

Keywords: game-theoretical semantics, IF logic, Monty Hall, dynamic epistemic logic, conditional probabilities

1 Monty Hall: Formulation of the problem

There are two formulations of the puzzles. The first one is more general:

> Monty Hall shows the contestant C three closed doors: behind one of them there is a prize, the other two are empty. C chooses a door. Monty Hall opens any of the other doors, which is empty. Then she asks C whether he would like to switch the doors, and choose the remaining one which is closed. Is it in C's interest to do it? (Richard Isaac, *The Pleasures of Probability* 1995, 3)

The second formulation mentions a particular door chosen by the contestant:

> Monty Hall (MH) hides a prize behind one of three doors, door 1, door 2, and door 3. The Contestant (C) has to guess it. Suppose his guess is door 1. Monty Hall, who knows the location of the prize and will not open

[1] I am indebted to Antonina Nepejvoda for the supercompilation of the Monty Hall.

> that door, opens door 3 and reveals that there is no prize behind it. She then asks C whether he wishes to change from his initial guess to Door 2. Will changing to door 2 improve C's chances of winning the prize? (Grinstead and Snell*, Introduction to Probabilities*, 1998)

The second formulation of the problem 'asks for the *conditional probability* that C wins if she switches doors, *given that* she has chosen door 1 and that Monty Hall has chosen door 3' (Grinsteas and Snell). On the other side, the first formulation is about the comparative probabilities of two kinds of *strategies* for C, the 'switch' strategy and the 'stay' strategy:

> We say that C is using the 'stay' strategy if she picks a door, and, if offered a chance to switch to another door, declines to do so (i.e., he stays with his original choice). Similarly, we say that C is using the 'switch' strategy if he picks a door, and, if offered a chance to switch to another door, takes the offer. Now suppose that C decides in advance to play the 'stay' strategy. Her only action in this case is to pick a door (and decline an invitation to switch, if one is offered). What is the probability that she wins a car? The same question can be asked about the 'switch' strategy. (Grinstead and Snell*, Introduction to Probabilities*, p. 137)

It should come as no surprize that the second formulation lends itself naturally to a solution in terms of conditional probabilities and updates in dynamic epistemic logic (DEL). The first formulation, on the other side, suggests a game-theoretical solution. I will give one.

2 The conditional probabilities account

We consider the second variant of the puzzle. We start with some abbreviations: D_i is going to abbreviate 'The prize is behind Door i'; B will be an abbreviation for 'Monty Hall opens door 3'. We make the initial assumption that there is an equal probability that the prize is between each of the three doors. Hence

$$P(D_1) = P(D_2) = P(D_3) = 1/3.$$

The second assumption is that when she has a choice, Monty Hall opens at random one of the two doors. In our particular situation in which C chose door 1, Monty Hall opens at random door 2 or door 3. Hence $P(B) = 1/2$.

The other probabilities are calculated as follows.

- When D_1, Monty Hall is free to open door 2 or door 3: $P(B/D_1) = 1/2$,
- When D_2, Monty Hall has to open door 3: $P(B/D_2) = 1$
- When D_3, Monty Hall has to open door 2: $P(B/D_3) = 0$.

In order to solve the puzzle, we have to calculate three conditional probabilities:

a) $P(D_1/B)$: the probability that the prize is behind door 1, given that Monty Hall opened door 3

b) $P(D_2/B)$: the probability that the prize is behind door 2, given that Monty Hall opened door 3

c) $P(D_3/B)$: the probability that the prize is behind door 3, given that Monty Hall opened door 3.

Using Bayes' theorem

$$P(A/B) = (P(B/A) \times P(A))/(P(B))$$

we obtain

$$\begin{aligned}
P(D_1/B) &= \frac{P(B/D_1) \times P(D_1)}{P(B)} = {}^1\!/_3 \\
P(D_2/B) &= \frac{P(B/D_2) \times P(D_2)}{P(B)} = {}^2\!/_3 \\
P(D_3/B) &= \frac{P(B/D_3) \times P(D_3)}{P(B)} = 0
\end{aligned}$$

Thus the answer to the initial question is: Yes, C should switch to door 2.

3 Conditional probabilities: trees

In order to facilitate comparison, we shall present the same solution given above using trees. (This is also the solution in Grinstead and Snell 1998.) The tree will consist of 12 branches which corresponds extensionally to all the possible choices of Monty Hall and the Contestant. Each maximal branch has the form (x, y, z), where:

- x stands for the door with the prize,
- y stands for the door chosen be C, and
- z stands for the door opened by Monty Hall.

In addition we have the following restrictions:

- If $x = y$, then z takes two possible values; and
- If $x \neq y$, then z can take only one value.

Thus the sequence $(1, 2, 3)$ represents the history:

1. MH hides the prize behind door 1; C chooses door 1; MH opens door 3.

It is customary in this setting to represent events as sets of branches of the tree. For instance, the event C_1 of C's choosing door 1 corresponds to

$$C_1 = \{(1, 1, 2), (1, 1, 3), (2, 1, 3), (3, 1, 2)\},$$

the event B of Monty Hall's opening door 3 corresponds to

$$B = \{(1, 1, 3), (1, 2, 3), (2, 1, 3)\},$$

and the event $C_1 \cap B$ of C's choosing door 1 and Monty Hall opening door 3 corresponds to

$$C_1 \cap B = \{(2, 1, 3), (1, 1, 3)\}.$$

Next, we endow the tree with a probability structure. First, we make the same assumptions as earlier:

- The events of the car being hidden behind door 1, door 2, and door 3 are equiprobable
- The events of C's choosing door 1, door 2, and door 3 are equiprobable.
- Whenever Monty Hall has a choice to open one of two doors, she chooses at random; and when she can open only one door, the probability is 1.

We can now calculate the probabilities of the events which interest us.

- The probability of the event $C_1 \cap B = \{(2,1,3),(1,1,3)\}$:

$$\begin{aligned} P(\{(2,1,3),(1,1,3)\}) &= \\ P(2,1,3)+P(1,1,3) &= \\ (1/3 \times 1/3 \times 1)+(1/3 \times 1/3 \times 1/2) &= 1/6 \end{aligned}$$

- The probability of the event $D_1 \cap C_1 \cap B = \{(1,1,3)\}$:

$$P(D_1 \cap C_1 \cap B) = 1/3 \times 1/3 \times 1/2 = 1/18 = 1/18$$

- The probability of the event $D_2 \cap C_1 \cap B = \{(2,1,3)\}$:

$$P(D_2 \cap C_1 \cap B) = 1/3 \times 1/3 \times 1 = 1/9 = 1/9$$

Finally, we apply Bayes' law to compute the probability that the car is behind door 1 given that C chose door 1and Monty Hall opend door 3, $P(D_1/C_1 \cap B)$, and the probability that the car is behind door 2 given that C chose door 1and Monty Hall opend door 3, $P(D_2/C_1 \cap B)$. We have

$$\begin{aligned} P(D_1/C_1 \cap B) &= \\ P(D_1 \cap C_1 \cap B)/P(C_1 \cap B) &= 1/18/1/6 = 1/3 \end{aligned}$$

and

$$\begin{aligned} P(D_2/C_1 \cap B) &= \\ P(D_2 \cap C_1 \cap B)/P(C_1 \cap B) &= 1/9/1/6 = 2/3. \end{aligned}$$

We have obtained the same solution that above.

4 Dynamic (Update) logic

4.1 Product updates

We consider the formulation to Monty Hall in which C chooses door 1. This carves out a subtree from the big tree above which consists of four maximal branches:

$$O_1 = (1,1,2)$$
$$O_2 = (1,1,3)$$
$$O_3 = (2,1,3)$$
$$O_4 = (3,1,2)$$

In update logic this tree is seen as generated in three stages.

1. First MH put the prize behind one of the three doors. This generates an epistemic model M_1 which corresponds to the first layer in the tree.

2. M_1 is then updated with $C's$ action a_1: C chooses door 1. The result is the product model M_2 which corresponds to the second layer.

3. Finally MH (publicly) opens some door. This updates M_2 with two possible actions, a_2(= she opens door 2), and a_3(= she opens door 3). The result is the product model M_3which corresponds to the third layer of the tree.

Each action is associated with a set of preconditions which specify in which circumstances (possible worlds) it may be performed. C's and Monty Hall's actions are governed by the following principles which determine their preconditions:

a) C may choose any of the three doors

b) Monty Hall can open only a door that$C$ did not choose, and where the car is not hidden.

Now some of the details.

The epistemic model M_1 has the form

$$M_1 = (W_1, R_C^1, R_{MH}^1)$$

where

- $W_1 = \{w_1, w_2, w_3\}$, (w_1 represents the world where the car is behind door 1, etc)
- $R^1_{MH} = \{(w, w) : w \in W_1\}$ (Monty Hall's actions are accessible to herself)
- $R_C = W_1 \times W_1$ (Monty Hall's actions are not accessible to the contestant C).

At stage (2), M_1 is updated with the action model $A_1 = (V_1, Q^1_C, Q^1_{MH})$, where $V_1 = \{a_1\}$. Given that $a_1$is a public action, both accessibility relations Q^1_C and Q^1_{MH} are $V_1 \times V_1$. From (a) we know that $Pre(a_1) = W_1$. Hence

$$M_2 = M_1 \times A_1 = (W_2, R^2_C, R^2_{MH})$$

where

- $W_2 = W_1 \times V_1$.
- $R^2_C = W_2 \times W_2$ (all the worlds in W_2 remain indistinguishable to C)
- $R^2_{MH} = \{(w, a_1), (w, a_1) : w \in W_1\}$ (Monty Hall knows exactly where she is).

Let us abbreviate the possible worlds in W_2 by:

$$\begin{aligned} v_1 &= (w_1, a_1) \\ v_2 &= (w_2, a_1) \\ v_3 &= (w_3, a_1) \end{aligned}$$

Finally, the product model M_2 is updated with the action model

$$A_2 = (V_2, Q^2_C, Q^2_{MH})$$

where

- $V_2 = \{a_2, a_3\}$, (a_2= Monty Hall opens door 2, etc).
- $Q^2_C = Q^2_{MH} = \{(a_2, a_2), (a_3, a_3)\}$.

From (b) we know that $Pre(a_2) = \{v_1,v_3\}$ and $Pre(a_3) = \{v_1, v_2\}$. The result of the update is the product model

$$M_3 = M_2 \times A_2 = (W_3, R_C^3, R_{MH}^3)$$

where $W_3 = W_2 \times A_2$ and the accessibility relations R_C^3 and R_{MH}^3 inherit the uncertainities from M_2.

Let us abbreviate the possible worlds of W_3 by:

$$\begin{aligned} x &= (w_1, a_1, a_2) \\ y &= (w_1, a_1, a_3) \\ z &= (w_2, a_1, a_3) \\ u &= (w_3, a_1, a_2) \end{aligned}$$

In order to give a solution to the puzzle, we need to establish what C knows at this stage, i.e. R_C^3. Given that a_2 and a_3 are public actions, C knows, after a_2 is performed, that she could be either in x or in u, i.e. R_C^3xu and R_C^3ux (plus the corresponding reflexivity conditions). And after a_3 is performed, she knows she can be either in y or in z, that is, R_C^3yz and R_C^3zy (plus the corresponding reflexivity conditions). Graphically:

$$\begin{array}{ccccccccc}
 & & w_1 & \cdots & \cdots & & w_2 & \cdots & w_3 \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & a_1 & \cdots & \cdots & & a_1 & \cdots & a_1 \\
 & \swarrow & & \searrow & & & \downarrow & & \downarrow \\
a_2 & & & & a_3 & \cdots & a_3 & & a_2 \\
x & & & & y & & z & & u
\end{array}$$

4.2 Product updates with probabilities

Earlier on, we endowed trees with a probability structure. We now do the same for product update models. I follow very closely van Benthem (2003).

For epistemic models M, we consider, for each agent i, the equivalence classes $D_{i,s} = \{t : R_i st\}$. Probability functions $P_{i,s}$ are defined over the probability space $D_{i,s}$. For simplicity, we take these functions to be uniform: all the worlds in the set $D_{i,s}$ are equiprobable. Following van Benthem, we simplify matters even more in finite models and assume that the functions $P_{i,s}$ assign probabilities

$P_{i,s}(w)$ to single worlds w. We can then use sums of these values to assign probabilities to propositions, viewed as the set of worlds where they are true. Then we can interpret $P_{i,s}(\varphi)$ as assigning a probabilistic value assigned to φ by the agent i in the possible world s. In case this value is 1, this will correspond to the assertion $K_i\varphi$.

Next, we assign probabilities to actions in the universe of the action models A. This is done relatively to a state s. The basic notion is $P_{i,s}(a)$: the probability that the agent i assigns to action a in the world s. In our example we assume that all this has been settled in some way or another, giving us agents' probabilities for worlds, and also for actions at worlds.

Finally we are ready to handle the puzzle. We are interested in the last update. Given that Monty Hall's action of opening a door is a public one, reference to the agent i does not matter, and we shall be concerned with probabilities functions of the form $P_s(a)$. We are interested in computing the relevant probabilities in

$$M_3 = M_2 \times A_2 = (W_3, R_C^3, R_{MH}^3).$$

More specifically, we are interested in the probabilities the agents assign to the possible worlds in W_3. As mentioned, these worlds have the form $v = (w_1, a_1)$, etc.

The central notion is $P_{c,(v,a)}(v', b)$: the probability agent C assigns to the world (v', b) in the world (v, a). In order to compute it, we need to know the probability $P_{i,v}(v')$ that C assigns to the world v' in v, and the probability $P_{v'}(b)$ assigned to the action b in the world v'. But this is not enough, for the action b could have been performed from any other world u indistinguishable (for agent C) from v. So we also need the probabilities $P_{C,v}(u)$ for every u such that $R_C vu$ together with the probabilities $P_u(b)$. Then we use the formula:

$$P_{C,(v,a)}(v', b) \quad = \quad \frac{P_{C,v}(v') \times P_{v'}(b)}{\sum_{R_C vu} P_{C,v}(u) \times P_u(b)}$$

Thus in our case we need to compute the value of

$$P_{C,v_1}(v_1) = P_{c,(w_1,a_1)}(w_1, a_1)$$

and that of

$$P_{C,v_1}(v_2) = P_{c,(w_1,a_1)}(w_2, a_1).$$

We have

$$\begin{aligned}
P_{c,(w_1,a_1)}(w_1, a_1) &= \\
\frac{P_{C,w_1}(w_1)\times P_{w_1}(a_1)}{P_{C,w_1}(w_1)\times P_{w_1}(a_1)+P_{C,w_1}(w_2)\times P_{w_2}(a_1) + P_{C,w_1}(w_3)\times P_{w_3}(a_1)} &= \\
\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times 1+\frac{1}{3}\times 1} &= \frac{1}{3}.
\end{aligned}$$

A similar computation yields

$$P_{C,v_1}(v_2) = P_{C,(w_1,a_1)}(w_2, a_1) = \frac{1}{3}$$

Finally we are interested in

$$P_{C,x}(y) = P_{C,(v_1,a_3)}(v_1, a_3)$$

and

$$P_{c,x}(z) = P_{c,(v_1,a_3)}(v_2, a_3).$$

The first one represents the probability that C assigns in the (actual) world (v_1, a_3) (the prize is behind door 1, C chooses door 1, Monty Hall opens door 3) to the very same world; the second one represents the probability that C assigns in the world (v_1, a_3) to the world (v_2, a_3) which is identical to the actual world, except for the prize being behind door 2.

We have

$$\begin{aligned}
P_{C,(v_1,a_3)}(v_1, a_3) &= \\
\frac{P_{C,v_1}(v_1)\times P_{v_1}(a_3)}{P_{C,v_1}(v_1)\times P_{v_1}(a_3)+P_{C,v_1}(v_2)\times P_{v_2}(a_3)} &= \\
\frac{\frac{1}{3}\times \frac{1}{2}}{\frac{1}{3}\times \frac{1}{2}+\frac{1}{3}\times 1} &= \frac{1}{3}
\end{aligned}$$

Similarly

$$\begin{array}{rcl} P_{C,(v_1,a_3)}(v_2,a_3) & = & \\ \dfrac{P_{C,v_1}(v_2) \times P_{v_2}(a_3)}{P_{C,v_1}(v_1) \times P_{v_1}(a_3) + P_{C,v_1}(v_2) \times P_{v_2}(a_3)} & = & \\ \dfrac{\frac{1}{3} \times 1}{\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1} & = & \dfrac{2}{3} \end{array}$$

We recover the same result as earlier: it is rational for C to switch to door 2.

5 Game-theoretical solutions

We consider the second formulation of the puzzle. To my knowledge, there is no full-fledged game-theoretical solution in the literature.

I will first describe a solution, due to Isaac (1995), which comes close to a game-theoretical one.

5.1 Isaac' solution

Isaac represents the puzzle as consisting abstractly of the succession of three actions:

a) C chooses one of the three doors

b) Monty Hall opens one of the two remaining doors, the one without a prize

c) C switches doors

followed by an a label W or L which shows whether C won or lost. The door where the prize is hidden is denoted by 1, the other two by 2 and 3.

Thus the sequence $(1, 2, 3, L)$ should be read:

> C chooses the door where the prize is; MH opens the other door 2; C switches to door 3; C looses.

Notice that the stage in the puzzle which indicates where Monty Hall has hidden the prize, is not explicitly represented. On the other side, there is one an extra-layer which represents C's action of switching doors and another extra-layer which specifies who lost or won. Notice also that the labels $1, 2$ and 3 are not rigid, they do not designate any concrete door.

When we think of C's action of switching doors, 4 possible situations can occur:

- (C chose the door where the prize is; Monty Hall opens the other door, 2; C switches doors; C looses): $(1, 2, 3, L)$
- Identical with the previous one, except that the last two choices are reversed: $(1, 3, 2, L)$
- (C chose one of the doors without the prize, say 2; Monty Hall opens the other door without the prize, 3; C switches to 1; C wins): $(2, 3, 1, W)$
- Identical with the previous one, except that the first two choices are reversed: $(3, 2, 1, W)$

We now have to endow the space

$$\{(1, 2, 3, L), (1, 3, 2, L), (2, 3, 1, W), (3, 2, 1, W)\}$$

with probabilities.

It is reasonable to assume that the probability that C chose the door where the prize is equals the probability that he chooses the door 2 (without the prize), and the probability that he chooses door 3. The event corresponds to

$$\{(1, 2, 3, L), (1, 3, 2, L)\}$$

and the last two ones to $\{(2, 3, 1, W)\}$ and $\{(3, 2, 1, W)\}$. So we assume that

$$\begin{array}{rcl} P(\{(1, 2, 3, L), (1, 3, 2, L)\}) & = & 1/3 \\ P(\{(2, 3, 1, W)\}) & = & 1/3 \\ P(\{3, 2, 1, W)\} & = & 1/3 \end{array}$$

We do not know the probabilities $P(1, 2, 3, L)$ and $P(1, 3, 2, L)$ but we shall not need them. What we are interested in is the event 'C wins' which corresponds to

$$\{(2, 3, 1, W), (3, 2, 1, W)\}$$

and the event 'C looses' which actually turns out to be the same event as 'C chose the door where the prize is'. Obviously

$$\begin{aligned} P(\{(2,3,1,W),(3,2,1,W)\}) &= \\ P(\{(2,3,1,W)\}) + P(\{(3,2,1,W)\}) &= 2/3 \end{aligned}$$

We now have an answer to our initial puzzle. Using the 'switching' strategies C will win with probability $2/3$ and loose with probability $1/3$.

A similar argument will represent the 'stick to the same door' strategy by

$$\{(1,2,1,W),(1,3,1,W),(2,3,2,L),(3,2,3,L)\}$$

By an argument similar to the previous one, we see that the probability that C wins ($= \{(1,2,1,W),(1,3,1,W)\}$) is $1/3$ whereas the probability that C looses is $2/3$.

Isaac's conclusion is: switching doors gives C a probability of $2/3$ to win the car, whereas sticking to his initial choice will give him a probability of $1/3$ to win the car.

Notice that:

- The solution is general, it concerns the first variant of Monty Hall puzzle.
- The solution does not appeal to conditional probabilities.
- There is a layer in the representation which makes explicit C's second guess.

We shall incorporate these elements in our game-theoretical solution.

5.2 A game-theoretical solution

We shall formulate Monty Hall as an extensive, finite win-loss game of imperfect information played by two players: the contestant C tries to identify the door with the prize whereas his opponent Monty Hall tries to deceive him. The game tree will extend the tree we introduced in connection with the conditional probabilities approach. Maximal branches will have now the form (x, y, z, t) with an extra term t to stand for the final choice of C. In this setting the maximal sequence $(1, 1, 2, 1)$ represents the possible play of the game:

> MH hides the prize behind door 1; C chooses door 1;
> MH opens door 2; C chooses door 1.

This play is a win for C if the last element of the sequence is the same as the first: in his second choice C chooses the door where the prize is. Note that in each play (x, y, z, t), x and z are choices made by Monty Hall, whereas y and t are choices made by C.

To specify the information of the players, notice that

C1 Any histories (x) and (x') are equivalent (indistinguishable) for player C

C2 Any histories (x, y, z) and (x', y', z') such that $y = y'$ and $z = z'$ are equivalent for player C.

(C1) tells us that C does not know the door where the prize is, when making his first choice. (C2) expresses the fact that C does not know the door where the prize is, when he makes his second choice.

Next we specify the strategies of the players.

We shall take the strategies of C to consist of pairs (f_y, f_t) of functions: $f_y$will give her a choice for y and f_t achoice for t. Given the requirement (C1), f_y will have to be a constant function, i.e. $f_y(x) = f_y(x')$ for any doors x, x' where Monty Hall hides the price. This amounts to f_y being an individual i (a door). Similar comments apply to f_t : given the requirement (C2), we can assume that f_t is a function h of two arguments, y and z. All in all we shall take C's strategies to consist of pairs (i, h_i), where i stands for a door and h_i for a function of two arguments (y, z).

A strategy (i, h_i) is winning if C wins every play where she follows it. The notion of 'following a strategy' is standard in game theory and we shall not give a formal definition.

We focuse on two kinds of strategies for player C (all the others are weakly dominated by them).

- The 'stay' strategy, S_C^{Stay}: choose a door, then stick to the initial choice no matter what Monty Hall does.

It is encoded by three strategy pairs, i.e.,

$$S_C^{Stay} = \{(i, h_i) : i = 1, 2, 3\},$$

where $h_i(y, z) = i$, for every y and z.

Each such strategy (i, h_i) is followed in every play

$$(x, i, z, h_i(i, z))$$

for any x and z. As mentioned earlier, it is winning whenever C's initial guess is correct, i.e., $i = x = h_i(y, z)$, and loosing otherwise. Obviously none of the 'stay' strategies is winning *simpliciter*.

- The 'switch' strategy, S_C^{Switch}: choose a door, and then after Monty Hall opens a door, switch doors.

This strategy is encoded by three strategy pairs

$$S_C^{Switch} = \{(1, f_1), (2, f_2), (3, f_3)\}$$

where

$$\begin{array}{ll} f_1(1,2) = 3 & f_1(1,3) = 2 \\ f_2(2,3) = 1 & f_2(2,1) = 3 \\ f_3(3,2) = 1 & f_3(3,1) = 2 \end{array}$$

Each of the three strategies wins in two cases: when the initial choice is incorrect, $i \neq x$; and it looses in one case, when the initial choice is correct.

Monty Hall's strategies consist of pairs (j, g): j is a value for x; and the function g associates to each argument (x, y) a value for z.

The only strategy available for Monty Hall (given the rules of the game) is: 'hide the prize behind a door, and after C chooses a door, open any other door'. Thus her set of strategies, S_{MH}, contains the following strategy pairs:

$(1, g_1)$:	$g_1(1,1) = 2$	$g_1(1,2) = 3$	$g_1(1,3) = 2$
$(1, g_1')$:	$g_1'(1,1) = 3$	$g_1'(1,2) = 3$	$g_1'(1,3) = 2$
$(2, g_2)$:	$g_2(2,1) = 3$	$g_2(2,2) = 1$	$g_2(2,3) = 1$
$(2, g_2')$:	$g_2'(2,1) = 3$	$g_2'(2,2) = 3$	$g_2'(2,3) = 1$
$(3, g_3)$:	$g_3(3,1) = 2$	$g_3(3,2) = 1$	$g_3(3,3) = 1$
$(3, g_3')$:	$g_3'(3,1) = 2$	$g_3'(3,2) = 1$	$g_3'(3,3) = 2$

Each of the strategy pair (j, g_j) is followed in every play of the form $(j, y, g_j(j, y), t)$, for any y and t. It is winning whenever $j \neq t$ and loosing otherwise. None of these strategies is winning simpliciter.

Monty Hall formulated as an extensive game of imperfect information is indeterminate: neither Monty Hall nor Eloise has a winning strategy.

To overcome indeterminacy we move to mixed strategies. Before, we need few definitions and results from classical game theory.

5.2.1 Strategic games: equilibria in pure strategies

A finite two player strategic game has the form $\Gamma = (S_I, S_{II}, u_I, u_{II})$ where:

1. S_I is the set of strategies of the first player
2. S_{II} is the set of strategies of the second player
3. u_I and u_{II} are the payoff functions of the players. That is, for every $\sigma \in S_I$ and $\tau \in S_{II}$, $u_I(\sigma, \tau)$ gives player I a payoff, which is a real number; and the same for u_{II}.

Fix a 2 player strategic game $\Gamma = (S_I, S_{II}, u_I, u_{II})$. When $\sigma^* \in S_I$ and $\tau^* \in S_{II}$, the pair (σ^*, τ^*) is an equilibrium in Γ iff the following two conditions are jointly satisfied:

(i) $u_I(\sigma^*, \tau^*) \geq u_I(\sigma, \tau^*)$ for every strategy σ in S_I. In other words

$$u_I(\sigma^*, \tau^*) = max_\sigma u_I(\sigma, \tau^*)$$

(ii) $u_{II}(\sigma^*, \tau^*) \geq u_{II}(\sigma^*, \tau)$ for every strategy τ in S_{II} . In other words

$$u_{II}(\sigma^*, \tau^*) = max_\tau \, u_{II}(\sigma^*, \tau)$$

When S_I and S_{II} are finite, there is a simple algorithm for identifying the equilibria:

- In each column, circle the maximum payoffs of player I (if the maximum payoff occurs more than once, circle every occurrence)

- In each row, circle the maximum payoffs of player II
- A pair of strategies (σ^*, τ^*) is an equilibrium in Γ iff both $u_I(\sigma^*, \tau^*)$ and $u_{II}(\sigma^*, \tau^*)$ are circled.

It is straightforward to transform the extensive Monty Hall game into a finite 2 player win-lose strategic game.

We shall take the two players to be Monty Hall and C.

We have already specified the set of strategies of Monty Hall, S_{MH}, and the set of strategies of the Contestant, S_C. Notice that whenever Monty Hall follows one of her strategies in S_{MH}, and C follows one of his strategies in S_C, a play of the extensive game is generated which is a win for either one of the players. For instance, when Monty Hall follows $(3, g_3)$ and C follows $(1, h_1)$, the result is the play $(3, 1, 2, 1)$ which is a win for Monty Hall. This will fix the payoff functions u_{MH} and u_C. Here is the matrix representation of the strategic Monty Hall game:

	$(1, g_1)$	$(1, g_1')$	$(2, g_2)$	$(2, g_2')$	$(3, g_3)$	$(3, g_3')$
$(1, h_1)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
$(2, h_2)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$
$(3, h_3)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
$(1, f_1)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0)$
$(2, f_2)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
$(3, f_3)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$

The rows represent the strategies of the Contestant and the colums those of Monty Hall. The reader may convince himself, by applying the algorithm described above, that there is no equilibrium in the game. This is, obviously, nothing else than the counterpart of the indeterminacy of the extensive game of imperfect information.

5.2.2 Strategic games: mixed strategy equilibria

Let $\Gamma = (S_I, S_{II}, u_I, u_{II})$ be a two player finite strategic game.

- A mixed strategy ν for player p is a probability distribution over S_p, that is, a function $\nu : S_p \to [0, 1]$ such that $\sum_{\tau \in S_p} \nu(\tau) = 1$.

- ν is uniform over $S'_i \subseteq S_i$ if it assigns equal probability to all strategies in S'_i and zero probability to all the strategies in $S_i - S'_i$.

Let $\Delta(S_p)$ be the set of mixed strategies over S_p. If $\mu \in \Delta(S_I)$ and $\nu \in \Delta(S_{II})$, the expected utility for player p is given by:

$$U_p(\mu,\nu) = \sum_{\sigma\in S_I}\sum_{\tau\in S_{II}} \mu(\sigma)\nu(\tau)u_p(\sigma,\tau).$$

We can identify a pure strategy $\sigma \in S_I$ with a 'degenerate' mixed strategy which assigns to σ probability 1 and 0 to all the other strategies in S_I. That is, when $\sigma \in S_I$ and $\nu \in \Delta(S_{II})$, we let

$$U_p(\sigma,\nu) \quad = \quad \sum_{\tau\in S_{II}} \nu(\tau)u_p(\sigma,\tau).$$

Similarly, when $\tau \in S_{II}$ and $\mu \in \Delta(S_I)$, we let

$$U_p(\mu,\tau) \quad = \quad \sum_{\sigma\in S_I} \mu(\sigma)u_p(\sigma,\tau).$$

Let $\Gamma = (S_I, S_{II}, u_I, u_{II})$ be a two player finite strategic game which is also a win-lose game (the only payoffs are 0 and 1). For $\mu^* \in \Delta(S_I)$ and $\nu^* \in \Delta(S_{II})$, the definition of (μ^*,ν^*) being a mixed strategy equilibrium in Γ is completely analogue to the earlier one.

The following results are well known.

Theorem 1 (von Neuman's Minimax Theorem). *Every finite, two-person, win-lose game has an equilibrium in mixed strategies.*

Corollary 1. *Let (μ,ν) and (μ',ν') be two mixed strategy equlibria in a win-lose game. Then $U_p(\mu,\nu) = U_p(\mu',\nu')$.*

The above results tell us that for two-player finite win-lose games an equilibrium always exists (von Neumann's theorem), and in addition, any two mixed strategy equilibria deliver the same expected utility. We shall take the value of the game to be the expected utility delivered by any of the mixed strategy equilibrium in the game.

We give a simple algorithm for identifying mixed strategy equilibria:

Proposition 1. *In a finite, two player strategic game, the pair* (μ^*, ν^*) *is an equilibrium if and only if the following conditions hold:*

1. $U_I(\mu^*, \nu^*) = U_I(\sigma, \nu^*)$ *for every* $\sigma \in S_I$ *in the support of* μ^*
2. $U_{II}(\mu^*, \nu^*) = U_{II}(\mu^*, \tau)$ *for every* $\tau \in S_{II}$ *in the support of* ν^*
3. $U_I(\mu^*, \nu^*) \geq U_I(\sigma, \nu^*)$ *for every* $\sigma \in S_I$ *outside the support of* μ^*
4. $U_{II}(\mu^*, \nu^*) \geq U_{II}(\mu^*, \tau)$ *for every* $\tau \in S_{II}$ *outside the support of* ν^*.

Here are few results from classical game theory which help us to reduce a game to a smaller one, after which we can apply the Proposition above.

Definition 1. Let $\Gamma = (S_I, S_{II}, u_I, u_{II})$ be a finite two player strategic, win-lose game. For $\sigma, \sigma' \in S_I$, we say that σ' weakly dominates σ if the following two conditions hold:

(i) For every $\tau \in S_I$:
$$u_I(\sigma', \tau) \geq u_I(\sigma, \tau)$$

(ii) For some $\tau \in S_{II}$:
$$u_I(\sigma', \tau) > u_I(\sigma, \tau).$$

A similar notion is defined for Abelard.

The following result enables us to eliminate weakly dominated strategies.

Proposition 2. *Let* $\Gamma = (S_I, S_{II}, u_I, u_{II})$ *be a finite 2 player, win-lose game strategic game. Then* Γ *has an equilibrium in mixed strategies* (μ_I, μ_{II}) *such that for each player* p *none of the strategies in the support of* σ_p *is weakly dominated in* Γ.

A proof of this fact may be found in Mann et al (Proposition 7.22).

Definition 2. Let $\Gamma = (S_I, S_{II}, u_I, u_{II})$ be a finite two player, win-lose strategic game. For $\sigma, \sigma' \in S_I$, we say that σ' is payoff equivalent to σ if for every $\tau \in S_{II} : u_I(\sigma', \tau) = u_{\exists I}(\sigma, \tau)$.

A similar notion is defined for Abelard. The next Proposition allows us to reduce the game to a smaller one by eliminating all the payoff equivalent strategies, except one.

PROPOSITION 3. *Let* $\Gamma = (S_I, S_{II}, u_I, u_{II})$ *be a finite two player, win-lose strategic game. Then* Γ *has an equilibrium in mixed strategies* (μ_I, μ_{II}) *such that for each player p there are no strategies in the support of* σ_p *which are payoff equivalent.*

A proof of this fact may be found in Mann et al (Proposition 7.23).

We now return to the strategic Monty Hall game. We notice that each strategy (i, h_i) is weakly dominated by some strategy (j, f_j). For instance $(1, h_1)$ is weakly dominated by $(2, f_2)$. Hence by the second proposition above we know that that game has the same value as the game

	$(1, g_1)$	$(1, g_1^{'})$	$(2, g_2)$	$(2, g_2^{'})$	$(3, g_3)$	$(3, g_3^{'})$
$(1, f_1)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0)$
$(2, f_2)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
$(3, f_3)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$

The next observation is that the strategies $(i, g_i^{'})$ and (i, g_i) are payoff equivalent for Abelard. Hence by the last proposition we know that the value of the game is the same as that of the game:

	$(1, g_1)$	$(2, g_2)$	$(3, g_3)$
$(1, f_1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
$(2, f_2)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
$(3, f_3)$	$(1, 0)$	$(1, 0)$	$(0, 1)$

Let μ be the uniform probability distribution, i.e. $\mu(1, f_i) = \dfrac{1}{3}$ and ν the uniform probability distribution $\nu(j, g_j) = \dfrac{1}{3}$. It is straightforward to check, using the first proposition above, that (μ, ν) is an equilibrium. The expected utility of player C (i.e., the value of the game) for this equilibrium is $2/3$.

Notice that C's strategy μ assigns an equal probability to each of the pure strategies which implement the 'switch' strategy. The important thing is not that it returns to player C an expected utility of $2/3$ but rather that it weakly dominates the 'stay' strategy. If we want to compute the expected utility returned to C by the latter strategy, we should return to the bigger game where both the 'switch' and the 'stay' strategies are listed. We know that the value of the game described there is the same as that delivered by the equilibrium pair (μ, ν). In that game, let ν^* be the same as ν, and let μ^* be the probability distribution such that: $\mu^*(i, f_i) = \dfrac{1}{3}$ and $\mu^*(i, h_i) = 0$. The pair (μ^*, ν^*) is an equilibrium in this larger game. We compute $U_C((i, h_i), \nu^*)$:

$$U_C((i,h_i),\nu^*) = \sum_{t \in S_{MH}} \nu^*(\tau) u_C((i,h_i),t) = 2 \times \frac{1}{6} \times 1 = \frac{1}{3}$$

In other words, the 'stay' strategy returns an expected utility of $\dfrac{1}{3}$.

We have obtained the same result as in Isaac's approach.

6 IF logic

IF logic (Independence-Friendly logic) is an extension of first-order logic which contains quantifiers and connectives of the form

$$(\exists x/W), \quad (\forall x/W), \quad (\vee/W), \quad (\wedge/W)$$

where the interpretation of e.g. $(\exists x/W)$ is: 'the choice of x is independent of the values of the variables in W'. When $W = \varnothing$, we recover the standard quantifiers. For illustration, the sentence

- For every x and x', there exists a y depending only on x and a y' depending only on x' such that $Q(x, x', y, y')$ is true

is rendered in the new symbolism by

$$\forall x \forall x' (\exists y/\{x'\})(\exists y'/\{x, y\}) Q(x, x', y, y').$$

IF sentences are interpreted by semantical games of imperfect information (Mann et al). However, we prefer to give an interpretation in terms of Skolem functions and Kreisel counterexamples.

The skolemized form or skolemization of φ, with free variables in U, $Sk_U(\varphi)$, is given by the following clauses:

1. $Sk_U(\psi) = \psi$, for ψ a literal
2. $Sk_U(\psi \circ \theta) = Sk_U(\psi) \circ Sk_U(\theta)$, for $\circ \in \{\vee, \wedge\}$
3. $Sk_U((\forall x/W)\psi) = \forall x Sk_{U\cup\{x\}}(\psi)$
4. $Sk_U((\exists x W)\psi) = Sub(Sk_{U\cup\{x\}}(\psi), x, f(y_1, ..., y_n))$

where $y_1, ..., y_n$ are all the variables in $U - W$ and f is a new function symbol of appropriate arity. We abbreviate $Sk_\varnothing(\varphi)$ by $Sk(\varphi)$.

Skolemizing makes explicit the dependencies of variables. We obtain an alternative definition of truth. For every IF formula φ, model $\mathbb{M}$, and assignment s which includes the free variables of φ we let: $\mathbb{M}, s \vDash^+_{Sk} \varphi$ if and only if there exist functions $g_1, ..., g_n$ of appropriate arity in M to be the interpretations of the new function symbols in $Sk_U(\varphi)$ such that

$$\mathbb{M}, g_1, ..., g_n, s \vDash Sk_U(\varphi)$$

where U is the domain of s. The functions $g_1, ..., g_n$ are called skolem functions.

We now define the dual procedure of Skolemization. The Kreisel form $Kr_U(\varphi)$ of the IF formula φ in negation normal form with free variables in U is defined by:

1. $Kr_U(\psi) = \neg\psi$, for ψ a literal
2. $Kr_U(\psi \vee \theta) = Kr_U(\psi) \wedge Kr_U(\theta)$,
3. $Kr_U(\psi \wedge \theta) = Kr_U(\psi) \vee Kr_U(\theta)$
4. $Kr_U((\exists x/W)\psi) = \forall x Kr_{U\cup\{x\}}(\psi)$
5. $Kr_U((\forall x/W)\psi) = Sub(Kr_{U\cup\{x\}}(\psi), x, g(y_1, ..., y_m))$

where $y_1, ..., y_m$ are all the variables in $U - W$.

We now obtain an alternative definition of falsity. For every IF formula φ, model $\mathbb{M}$, and assignment s which includes the free variables of φ we let: $\mathbb{M}, s \vDash^-_{Sk} \varphi$ if and only if there exist $h_1, ..., h_m$ in

M to be the interpretations of the new function symbols in $Kr(\varphi)$ such that

$$\mathbb{M}, h_1, ..., h_m, s \vDash Kr_U(\varphi)$$

where U is the domain of s. We call $h_1, ..., h_m$ Kreisel counterexamples.

The Monty Hall game is expressed in IF logic by the sentence

$$\forall x(\exists y/\{x\})\forall z[x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\})x = t]$$

or equivalently by the sentence φ_{MH}

$$\forall x(\exists y/\{x\})\forall z[x = z \vee y = z \vee (\exists t/\{x\})x = t].$$

We can think of the Contestant, C, as the existential quantifier and disjunction, and of Monty Hall as the universal quantifier. We do not want to push the formalization too far. The intuitive reading of our sentence should be clear: For all Door x where the prize is hidden by Monty Hall, for every door y guessed by C, for every door z opened by Monty Hall, if z is distinct from x and from y, then C has one more choice to identify the door where the prize is. The Skolemized form of φ_{MH} is

$$\forall x \forall z[x = z \vee c = z \vee x = f(c, z)]$$

and its Kreisel form is

$$\forall y[d = g(d, y) \vee y = g(d, y) \vee \forall t(g(d, y) = t)]$$

where c, d, f and g are new function symbols. The reader should convince herself that on models $\mathbb{M} = \{1, 2, 3\}$ (corresponding to the three doors) the possible values of (c, f) correspond to the set of strategies of the Contestant; and the possible values of (d, g) correspond to the set of strategies of Monty Hall.

Then we can identify the value of φ_{MH} in the model $\mathbb{M} = \{1, 2, 3\}$ to be $2/3$.

7 Conclusion

The updated account gave the same solution to the Monty Hall problem as the classical account based on conditional probabilities. Both approaches conditionalize, the former on actions, the second on propositions and yield two posterior probabilities: $P(D_1/B) = 1/3$ and $P(D_2/B) = 2/3$ in the latter; and $P_{c,(v_1,a_3)}(v_1, a_3)$ and $P_{c,(v_1,a_3)}(v_2, a_3)$ in the former. I take both approaches to provide a solution to a *particular, local, decision theoretical* problem, that of explaining why a particular action is more rational than another in certain particular circumstances.

Yet there are important differences between them. Van Benthem points out that the conditional probabilities account describes what would be the probability that the car is behind door 1 if B *were to happen* (alternatively, if action a_3 were to be performed). On the other side, he takes $P_{(w_1,a_1)}(\varphi)$ (reference to the agent C has been erased) to describe rather the probability of φ in the state (w_1, a_3) reached *now after action a_3 has been performed*. 'The [latter] takes place once arrived at one's vacation destination, the [former] is like reading a travel folder and musing about tropical islands. The two points are related, but not identical'.

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Semiotic foundations of logic

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ABSTRACT. The article offers a look at the combinatorial logic as the logic of signs operating in the most general sense. For this it is proposed slightly reformulate it in terms of introducing and replacement of the definitions.

Keywords: combinatory logic, semiotics, definition, logic foundations

1 Language selection

Let's imagine for a moment what would be like the classical logic, if we had not studied it in the language of negation, conjunction, disjunction and implication, but in the language of the Sheffer stroke. I recall that it can be defined with help of negation and conjunction as follows

$$A \mid B =_{Df} \neg(A \wedge B).$$

In turn, all connectives can be defined with help of the Sheffer stroke in following manner

$$\begin{gathered} \neg A =_{Df} (A \mid A) \\ A \wedge B =_{Df} (A \mid B) \mid (A \mid B) \\ A \vee B =_{Df} (A \mid A) \mid (B \mid B) \\ A \supset B =_{Df} A \mid (B \mid B). \end{gathered}$$

Modus ponens rule takes the following form

$$\frac{A,\ A \mid (B \mid B)}{B}.$$

We can go further and following the ideas of M. Schönfinkel to define two-argument infix quantifier '$|^x$'

$$A \mid^x B =_{Df} \forall x(A \mid B).$$

Now we can use it to define Sheffer stroke and quantifiers.

$A \mid B =_{Df} A \mid^x B$ where the variable x is not free in the formulas A and B;
$\forall x A =_{Df} (A \mid^y A) \mid^x (A \mid^y A)$ where the variable y is not free in the formula A;
$\exists x A =_{Df} (A \mid^x A) \mid^y (A \mid^x A)$ where the variable y is not free in the formula A.

The rule for the introduction of the universal quantifier takes the form

$$\frac{\vdash A}{\vdash (A \mid^y A) \mid^x (A \mid^y A)}.$$

The language containing the only quantifier '$|^x$' is functionally complete, and has the same expressive power as the language of the classical predicate logic.

Imagine now that theorems of *Principia Mathematica* are formulated and proved in this language, and that all the fundamental theorems of logic, arithmetic and set theory are described in the language.

We would have the same results as today, but it would be difficult to convince other people that what we learn is really logic. In response, we probably would have heard that we have created an interesting mathematical tool, but it has little to do with logic.

2 Combinatory logic

Something similar has happened to the combinatory logic, which was born December 7, 1920, when M. Schönfinkel has made his now famous report to the Mathematical Society of Göttingen. In this report Schönfinkel [4] has showed that not only logical connectives but also quantifiers can be reduced to a single two-place operation.

He also showed that under the assumption that functions themselves can serve as arguments to other functions and to be their values, we can get along one-argument functions, and two operations to combine them, which can be summarized as follows

$$\mathbf{K}xy = x \text{ and } \mathbf{S}xyz = xz(yz).$$

The role of **K** and **S** in the language is similar to that of the connectives in the logic, but the fundamental difference is that they are applicable to expressions that represent the objects of any nature (!) rather than for solely sentences.

With these operations, which are called combinators, we can define any other operations with functions including quantifier '$|^x$'. In this sense, the set of **K** and **S** is complete. The theory of combinators is called combinatory logic.

Alphabet

1. Var — a set of variables;
2. **K** and **S** — constants;
3.) , (— brackets;
4. $\geqslant$ — reduction character.

Terms

1. All $x \in Var$ are terms;
2. **K** and **S** are terms;
3. If X and Y are terms, then (XY) is a term;
4. Nothing else is a term.

Reductions

1. If X and Y are terms, then $X \geqslant Y$ is reduction;

2. Nothing else is reduction.

Axioms

A.1 $X \geqslant X$

A.2 $\mathbf{K}XY \geqslant X$

A.3 $\mathbf{S}XYZ \geqslant XZ(YZ)$

Rules

R.1 $X \geqslant Y \implies XZ \geqslant YZ$

R.2 $X \geqslant Y \implies ZX \geqslant ZY$

R.3 $X \geqslant Y,\ Y \geqslant Z \implies X \geqslant Z$

We are not going to develop in detail the combinatory logic and prove metatheorems related to it.

Unfortunately, due to the high degree of abstraction of the combinatory logic, it is not widespread, although many logicians have heard of its existence. The combinatory logic is much more known to specialists in computer science, which refer to it as a mathematical apparatus of functional programming. It is not considered as a theory of correct reasoning. It seems unclear how to use it to analyze usual reasoning. There were many attempts to synthesize the combinatory logic with the logic in the traditional sense, but as a result received either contradictory logical systems, or systems that have not received wide acceptance and recognition.

3 The fundamental nature of the combinatory logic

The main obstacle to widespread use of the combinatory logic to analyze the reasoning lies in the highly abstract nature of the basic combinators, and hence the lack of understanding of how to make their use in natural discourse. The situation is somewhat similar to that if we developed a logic-based language with the single Sheffer stroke.

However, the combinatory logic is self-sufficient. There is no need to go beyond it, to present it as the logic of constructing arguments. It's enough just to reformulate it a bit.

First of all it is necessary to give up some of the stereotypes. For a long time it was a stereotype of logic as a theory about relationships in the sphere of common terms. It was a characteristic of the Aristotelian approach, which has dominated for over two thousand years. G. Frege has refused the stereotype and begun to examine the logic as a theory of propositional functions. According to his words, the logic is a theory of Truth Being.

This view turned out to be very fruitful, and we still are under its influence. The theory of combinators does not fit neither Aristotlian, nor Fregean approach, since appeals to a more fundamental entities than the common terms and propositions. It seems to me, in this case we are dealing with the logic of signs operating in the most general sense.

'*Combinatory logic is a branch of mathematical logic which concerns itself with the ultimate foundations. Its purpose is the analysis of certain notions of such basic character that they are ordinary taken for granted*' [3, p. 1].

The expressions of language, considered as signs, act as representatives of the various objects of thought, which can be things, properties, functions, relationships, etc. Assignment of thought objects to the specific categories is possible, but it isn't a problem of logic, since it can happen only as a result of later cognitive activity of the subject of cognition. The logic is a servant of science, not trendsetter.

The power of language as an instrument of cognition is that it allows us to manipulate objects of extralinguistic reality on the sign level. This manipulation can be represented as a chain of transitions from one signs to another. In the process of manipulating signs the relationship between them and the objects of external reality should not be lost. This is a necessary condition for the correctness of our cognitive activity. Some transitions between signs can be justified by the already known properties of the objects of thought correlated to them. Other transitions are due to study the intrinsic properties of sign systems. In a sufficiently general form the consequence relation between signs can be defined as follows:

U follows from the premisses $\Sigma = \{V_1, \ldots, V_n\}$, if and only if there exists a rule R, which allows on the basis of the values of premises Σ to determine the value of the expression U.

Formulated in this way the idea of reasoning does not need to clarify what specific linguistic expressions are used, what is the nature of the correspondence between these expressions and extralinguistic reality, what exactly is this reality.

4 What is the rule R?

In the classical logic, where the premises and conclusion are sentences, this rule is specified as '*if the premises $V_1, \ldots, V_n$ are true, then the conclusion U is true*'. Obviously, in this case, the rule R is a partial function that is defined only when all the premises are true. If at least one of the premises is false, we can not say with certainty what will be the truth value of the conclusion. In this case the problem is complicated by the fact that for sufficiently rich theory it is fundamentally impossible to prove that the theory is consistent, and to prove that there is at least one model that makes all the premises true. The standard definition of the semantic consequence stops working.

Our definition does not require that all the premises were true. We can build a system of reasoning in which the sentence *not*-A follows from the sentence A. Indeed, if we know the truth-value of the proposition A, then there is no problem to find the truth-value of proposition *not*-A. When you're trying to explain it to professional logician, you will often come across a misunderstanding and opposition, the reason of which are stereotypes inherited from the classical logic approach.

Our definition does not demand that assumptions and conclusions necessarily have been sentences. If you take the language of arithmetic, the formula $t_1 = t_2$ follows from two terms t_1 and t_2, because, knowing the values of arithmetic terms t_1 and t_2, we can always determine the truth value of the formula $t_1 = t_2$. Similarly, the term t_2 follows from the terms t_1 and $t_1 + t_2$. In the last example we have the consequence relation between the terms, not sentences.

Inference rules of any system of logic, that we use during construction of the syntactic metalanguage proofs, are also examples of rule R from our definitions of consequence.

The rule R can be a computer program, that from the input parameters (premisses) calculates the result (conclusion), which can be a solution of the differential equation, a corrected word from a text, an informative chart, a piece of music, characters that appear on the screen depending on what keys are pressed, etc.

A student of Architectural Institute in defending diploma must convince the commission of examiners that if you bring sand, water, cement and bricks, and then follow the course of action to certain rules and drawings of the project, the result will really skyscraper, and not another wreck.

A person who is engaged in guesswork, also follows the rules associated with some sign system. Astrologers, African shamans, fortune-tellers on the cards use their sign systems and their own rules relating assumptions and conclusions.

In natural language sentence '*Mary loves Bob*' follows from the words '*Bob*', '*Mary*', '*love*'. This following takes place quite independently of whether or not Mary loves Bob. It takes place because of our knowledge of the rules of morphology, grammar and semantics of English. Even if we are for the first time in our lives hear or read this sentence, but if the values of the words '*Bob*', '*Mary*' and '*love*' are known to us, thanks to our knowledge of the rules of language, we always can define the value of the sentence '*Mary loves Bob*'.

5 Formalism of signs

As can be seen from the above examples our definition covers a fairly wide range of phenomena. It is not limited to sentences, but is applicable to the signs of a different nature. In order to cover it in one logical formalism, we must find a starting point to represent the signs.

To do this, I recall the words of a well-known linguist Emile Benveniste that '*... language has a configuration in all its parts and as a totality. It is in addition organised as an arrangement of distinct* *and distinguishing 'signs', capable themselves of being broken down into inferior units or of being grouped into complex units. This great structure, which includes substructures of several levels, gives its form to content of thought... Linguistic form is not only the condition for transmissibility, but first of all the condition for the realization of thought. We do not grasp thought unless it has already been adapted to the framework of language*' [1, pp. 55–56].

The basic idea is that from the syntactic point of view signs form a hierarchy. Complex symbols are obtained by combining simpler ones. To do this, we can use a number of different brackets.

For example, a pair of brackets $\langle \_, \_, \_ \rangle^a$ may be three-argument operation, which allows us to construct complex sign $\langle$'*apple*', '*is*', '*red*'$\rangle^a$ from three words '*apple*', '*is*' and '*red*'. From signs '*sky*', '*is*' and '*blue*' we can construct a new sign $\langle$ '*sky*', '*is*', '*blue*' $\rangle^a$. Other types of brackets will be needed to build such signs as $\langle$'+', '3', '2' $\rangle^b$ and $\langle$ '*young*', '*man*' $\rangle^c$.

It is easy to show that in fact we need only one pair of brackets, which is applicable only to pairs of signs. That's how we do that, because it is convenient for the demonstration of connection with the combinatory logic, but application tasks may require different sets of brackets [5].

Alphabet

1. Var — a set of variables.
2. $Const$ — possibly empty set of constants;
3. $), ($ — brackets.

Terms

1. 1. All $x \in Var$ are terms;
2. All $c \in Const$ are terms;
3. If U and V are terms, then (UV) is term;
4. Nothing else is a term.

For dropping brackets we accept an agreement about their association to the left.

From the algebraic point of view the models of this language are groupoids, which can be represented as follows

$$\mathbf{M} = \langle D, O, I \rangle$$

where D is nonempty set
$O : D \times D \to D$ — two-argument operation on D
I — a function for the interpretation of language constants.

Let $Val = D^{Var}$ — the set of all functions assigning values to variables. Then the value of the term U in the model $\mathbf{M} = \langle D, O, I \rangle$ for the evaluation $v \in Val$ is defined in the obvious way:

1. $\mid x \mid_v = v(x)$, if $x \in Var$;
2. $\mid c \mid_v = I(c)$, if $c \in Const$;
3. $\mid (UV) \mid_v = O(\mid U \mid_v, \mid V \mid_v)$.

Now we are ready to define rigorously the central concept of consequence $\Sigma \parallel\!= U$, where Σ is a finite set of terms, and U is a term.

Term U follows from the set of terms $\Sigma = \{V_1, \ldots, V_n\}$ if and only if for any model $\boldsymbol{M}$ there exists a function f, such that for every valuation $v \in Val$ holds $\mid U \mid_v = f(\mid V_1 \mid_v, \ldots, \mid V_n \mid_v)$.

$$\{V_1, \ldots, V_n\} \parallel\!= U \iff \forall M \exists f \forall v (\mid U \mid_v = f(\mid V_1 \mid_v, \ldots, \mid V_n \mid_v)).$$

It is easy to verify that this consequence relation satisfies the well-known conditions of Tarski:

1. If $U \in \Sigma$, then $\Sigma \parallel\!= U$;
2. If $\Sigma \parallel\!= U$ and $\Sigma \subseteq \Delta$, then $\Sigma \parallel\!= U$;
3. If $\Sigma \parallel\!= V$ and $\Sigma, V \parallel\!= U$, then $\Sigma \parallel\!= U$.

In addition, the defined relation is structural, i.e. $e(\Sigma) \parallel= e(U)$ follows from $\Sigma \parallel= U$, where e — is a substitution on set of terms of the language. It means that the consequence relation defines the logic in the sense of Tarski.

Let's turn to the typology of signs of Peirce–Morris. There are three groups of signs.

The first group — **signs-indices**, whose connection with the signified objects may be due to temporal, spatial and causal types of relationships. Analysis of specific types of relationships is beyond the scope of logic. Distinguishing feature of signs-indices is that they have no significant similarity with their objects, that they refer to individual things, to individual objects, to single sets of objects and direct our attention to their objects by blind compulsion. We can assume that in our formalism signs-indices are among initial constants of language.

The second group of signs — **iconic signs**. They are linked to the signified objects through the structural similarity. Charles S. Peirce believed that any algebraic equation is an iconic sign because it shows using algebraic symbols (which themselves are not iconic signs) the relationships between it's variables. Every formula of logic and every term can also be regarded as iconic signs. It is thanks to knowledge of their structure we have the ability to operate with them on the basis of formal rules. The situational logics explicitly use iconicity property of complex expressions of the language.

The third group of signs are **signs-symbols**, whose connection with the signified objects is quite arbitrary, it exists only for the language interpreter. By his request, he gives the role of some objects to be representatives of other objects. In logic the operation of introducing of signs-symbols is well-known and is called the operation of definition. In our language we can represent it as follows.

Suppose that we have in our language a term T, all the variables of which are contained among set of variables $\{x_1, \ldots, x_n\}$. Then we can add to our language a new constant $\mathbf{D}$, taking the following definition

$$\mathbf{D}x_1 \ldots x_n =_{def} T.$$

By adopting a definition the interpreter is able at any moment to take advantage of the inverse operation of substitution, or disclosure definition, and make the transition from the term $X\{\mathbf{D}Z_1 \dots Z_n\}$, in which there is occurrence of the term $\mathbf{D}Z_1 \dots Z_n$, to the term $X\{T[Z_1/x_1, \dots, Z_n/x_n]\}$, obtained as a result of replacement $\mathbf{D}Z_1 \dots Z_n$ with simultaneous substitution of terms $Z_1, \dots, Z_n$ in the term T instead of the variables $x_1, \dots, x_n$. We can summarize this as the next rule

$$\frac{\mathbf{D}x_1 \dots x_n =_{def} T, \quad X\{\mathbf{D}Z_1 \dots Z_n\}}{X\{T[Z_1/x_1, \dots, Z_n/x_n]\}}.$$

The reasoning in this logic can be defined as a sequence of terms $T_1, \dots, T_n$, each of which is either one of the premises, or obtained from the previous terms of the sequence by the rule of disclosure of definitions. We specially focus attention on the fact that the possible values of terms occurring in the derivation can be any objects, and not necessarily the truth values of sentences.

Basically, nothing new compared to existing combinatory logic has been proposed. In the combinatory logic we accept reductions $\mathbf{K}XY \geqslant X$ and $\mathbf{S}XYZ \geqslant XZ(YZ)$ as initial, and in our proposed formalism we can introduce definitions $\mathbf{K}XY =_{def} X$ and $\mathbf{S}XYZ =_{def} XZ(YZ)$ and get all the same. In this sense, the combinatory logic is embedded in the formalism we have constructed.

6 Comments

I. Prior to emergence of the science of logic, much attention was paid to the special operation on linguistic expressions, which later became known as the 'operation of definition'. The idea of this operation is transparent to the understanding and hardly anyone would reject it. You may recall that science begins not so much with theorem proving, as with search for definitions of the various objects and phenomena of reality. Plato's Socrates seeks definitions of beautiful jug, a beautiful woman, beauty by itself, courage, and many others. Thus he extended the language for the description of the world, which allowed him a new way to classify phenomena, to move to a higher level of abstraction and then to think in a new system of concepts. Modern science is unthinkable without an abstract and ideal objects, introduced by different kinds of definitions. Any system of axioms implicitly defines the terms imposed by it. The combinatory logic can be rigorously represented as a system of reasoning based on accepted definitions. In this case, its meaning is extremely clear. It is interesting that the initial set of constants *Const* may be empty, but it can not prevent a logical subject to start using the operation of definition and thus start forming a system of concepts, which are essentially a priori.

II. Initial Schönfinkel's combinators **K** and **S** are of interest because they represent the complete set of combinators, through which you can define any other combinators. But in the logic of reasoning based on the principle of introducing definitions and their replacement there is no need to declare some initial basic combinators. Through definitions, depending on a specific problem to be solved, you can introduce any desired combinators, not reduce them to a mandatory basis. This significantly simplifies many of constructions and reasoning, making them closer to the natural.

III. It is known that in the combinatorial logic it is provable theorem that each combinator **C** has a fixed point, i.e. there exists such a term T that $\mathbf{C}T = T$ is provable. Similarly, we can show that fixed-point definitions are allowed in the proposed formalism. That is, if there exists a term T, all the variables of which are contained in set of variables $\{y, x_1, \ldots, x_n\}$, then we can introduce the fixed-point definition

$$\mathbf{D}x_1 \ldots x_n =_{dfp} T\,[\mathbf{D}/y]\,.$$

These definitions give rise to a conservative extensions of the existing formalism, which at the same time are not inessential. This means that the system of logic, based on the principle of introducing definitions and their replacement which is supplemented by the fixed point definitions, has a greater expressive power than the combinatory logic, since combinatory logic can be embedded into it, but reverse isn't true. It is appropriate to draw an analogy between the constants, introduced by means of fixed point definitions, and ε-terms in the classical logic.

IV. It is known that all recursive functions can be represented in combinatory logic. They are also obviously representable in the proposed formalism. But the heart of this formalism is the logical operation of definition. Hence it follows that nature of computable functions is exclusively logical and does not depend on adoption any assumptions about the world around us. Since the proposed formalism has clear semiotic foundations and at the same time is closely connected with computability, it can be regarded as the perfect language, which G. Chaitin dreamed of [2].

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An approach to the interpretation on intensional contexts

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abstract. The paper introduces a non-standard analysis of intensional contexts on the ground of generalized approach to semantics construction. The principles of building such kind semantics are consider. As far as I can see it is an idea on domains and anti-domains that lays in the ground a semantics of intensional contexts. Intensional contexts differ from extensional by ascription of specific values to intensional predicates (operators) and, what is more important, by a way of their combination with arguments. Thus constructing operations play the leading role in proposed analysis. The peculiarities of IPL: any expression including intensional predicates and operators has an intension as well as an extension.

Keywords: generalized semantics, domains and anti-domains, propositional concept, operation of abstraction

The paper introduces a non-standard analysis of intensional contexts on the ground of generalized approach to semantics construction. In so doing, an expressive power of the natural language appropriate for the representation of intensional context's logical structure is considered.

A logical structure of an intensional context is determined by both interpretation of intensional signs and accepting a specific applicative operation of intensional operators (predicates) to terms. This procedure defines an algorithm, which allows finding extensions and intensions of corresponding contexts.

It is a set of laws, presuppositions and conventions Γ that being accepted determines states of affairs in possible worlds semantics.

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When epistemic contexts such as $\mathbf{B}a(p)$ considered, these principles and conventions depend on a subject a, denoted as Γa. They may be partially or completely agree with the laws of a theory, that is $\Gamma a \subseteq \Gamma$ or $\Gamma a \cap \Gamma$ and so on (where $\Gamma a = \Gamma$ is the case of omniscience). If $\Gamma a \neq \Gamma$, a subject can 'break the laws' of a theory, because they are not included in Γa. To this extent the subjective worlds WP$^{\circ}$ may be imaginary, and give rise to contradictions and paradoxes. This idea is to be taken into account when argumentation and the process of conviction are considered.

A semantical analysis of epistemic contexts generate a bulk of questions. What is possible interpretation of epistemic operators? What are the truth-conditions for epistemic statements with such operators? What is their logical structure? In what follows we will focus on a method of interpretation of intensional signs (operators and predicates) on the ground of a generalized approach to semantics construction.

We consider that adequate semantics may be constructed without using the concepts of contradictory or incomplete state descriptions. In any case, these concepts are not taken as a background and no assumptions are made in relation to the objects of discourse.

Instead partially defined predicates are accepted. We consider that predicates of truth, falsity belong to this kind — they can be partially defined. Second, we proceed from the idea of the symmetry of concepts of truth and falsity (and this is very important). Falsity is considered to be an independent notion and not as absence or negation of the truth.

Let us consider the principles of building language semantics. I shall construct my semantics using the notion of possible worlds. Let W be a non-empty set of possible worlds, φ a function ascribing a pair of sets $\langle H_1, H_2 \rangle$ to propositional variables where $H_1 \subseteq W$, $H_2 \subseteq W$.

$\varphi_T(p) = H_1$ is the class of worlds in which p holds (the domain of p).

$\varphi_F(p) = H_2$ is the class of worlds in which p does not hold (the anti-domain of p).

The function of ascribing values to propositional variables is given in a generalized form: not the truth values in a given world, that is, not the objects t and f, are ascribed to propositional variables, but special 'intensional objects' — classes of worlds $\varphi_T(p)$ and $\varphi_F(p)$. It is this that gives the intensional character to the propositional connectives, cf. [3, V].

We shall use a propositional language with the logical connectives &, $\vee$, $\supset$, $\sim$. Let us introduce conditions of ascribing truth values to complex formulas as follows:

$$\varphi_T(A \,\&\, B) = \varphi_T(A) \cap \varphi_T(B), \quad \varphi_F(A \,\&\, B) = \varphi_F(A) \cup \varphi_F(B),$$
$$\varphi_T(A \vee B) = \varphi_T(A) \cup \varphi_T(B), \quad \varphi_F(A \vee B) = \varphi_F(A) \cap \varphi_F(B),$$
$$\varphi_T(\sim A) = \varphi_F(A), \quad \varphi_F(\sim A) = \varphi_T(A).$$

When defining logical connectives, no limitations are imposed on the relations between the classes $\varphi_T(A)$ and $\varphi_F(A)$. The independence in ascribing domains and anti-domains to propositions allows us to treat the operation of negation in a generalized way. As a result of the above mentioned principles we get semantics with truth value gaps and with glut evaluations.

Dealing with such objects as the classes $\varphi_T(A)$ and $\varphi_F(A)$ it is possible to establish different relations between them, to accept or not to accept conditions (1) and (2). It is possible to accept one of them and reject the other, for they are independent of one another.

The relation between the classes $\varphi_T(A)$ and $\varphi_F(A)$ may but need not satisfy the following conditions:

(1) $\varphi_T(A) \cap \varphi_F(A) = \varnothing$, (2) $\varphi_T(A) \cup \varphi_F(A) = W$.

Accepting both (1) and (2) we get the standard semantics. Accepting (1) and rejecting (2) — shortly (1), $\overline{(2)}$ — semantics with truth value gaps; accepting (2) and rejecting (1) — semantics with glut evaluations (which permits of the overlap of truth and falsity); rejecting both (1) and (2) we get relevant semantics.

If both conditions (1) and (2) are accepted, the class of tautological formulas coincides with the class of irrefutable formulas and is identical to the class of tautologies of classical logic.

One of the peculiarities of analysis of intensional contexts is connected with an interpretation of intensional signs. This interpretation presupposes introduction of very special objects and leads to multiplication of abstract entities within semantical analysis. In other words this interpretation entails certain 'intensional' ontology. Let us examine what are the entities introduced this way. As far as I can see it is an idea of domains and anti-domains that lays in the ground of the semantics of intensional contexts.

A reference of an expression in a certain world represents its extension. If, following Carnap, we interpret intension as those entities that two L-equivalent statements A and B have in common, then they appear to be their domains: $\varphi_T(A) \equiv \varphi_F(A)$. An intension of a statement is often called its proposition or *propositional concept*.

Let $\mathbf{s}$ means a domain of a proposition — $\varphi T(A)$, and $\bar{\mathbf{s}}$ — its anti-domain, where $s \in 2^W$ and $s \subseteq W$.

Now we turn to a set h, whose elements are propositions — $h = \{\mathbf{s}_1, \ldots, \mathbf{s}_l\}$. Hence, the domains are represented by families of propositions. Consider possible interpretations of modal operators and intensional predicates. Let their references be functions or relations defined on domains of statements or on families of such domains. Then the assignments are as follows.

Let M be intensional operator (for example $\Box p$) of the type $s//s$, then possible referents are:

I $(2^W)^{(2^W)}$, where 2^W is a propositional concept (intension) — $\mathbf{s}$, i.e. $\varphi_T(p)$;

II $2^{W \times 2^W}$ — a relation G; $\langle w_i, \mathbf{s}_i \rangle \in G$.

If we want an intensional sign to have both intension and extension, then we chose

III $(2^{(2^W)})^W$, that is $f\colon W \to 2^{(2^W)}$, i.e. $w_i \to \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is an intension $\mathbf{M}$, while its extension in a world w_i is $f(w_i) = h$, that is a a set of propositional concepts, where $w_i \in s_i$; $s_i \in h$; $h \in 2^{(2^W)}$.

An interpretation I represents the approach of D. Scott, and an interpretation II corresponds to R. Montague. The main advantage of the IIId approach lies in possibility to assign not only intensions or extensions but both of them as well.

An interpretation of intensional predicates is analogous [3, p. 246].

Now consider a structure of intensional contexts. This structure is determined by an interpretation of intensional signs, and first of all by the type of intensional entities assigned.

Logical aspect of analysis of intensional contexts is important for us. A key to the puzzle of these contexts can be found just there not in behavior of proper names. The core idea of this approach is that semantical analysis of intensional contexts presupposes, first of all, identification of peculiarities of their logical structure. What is the way in which components are linked in contexts that contain intensional predicates and operators? Thus *constructing operations* play the leading role in proposed analysis. Let us construct a semantics for Intensional Predicate Logic (IPL).

The language of IPL is based on the theory of semantical categories but the notion of index of category is extended: (1) n and s are indexes of categories (n is a category of singular terms; s is a category of sentences); (2) if α and β are indexes of categories then α/β and $\alpha//\beta$ are indexes of categories. All categories of the type α/β are extensional and those of type $\alpha//\beta$ are intensional. The method of interpretation for these two types of categories is especially important.

Let a model structure be the construction $\langle W, N, U, I, \Psi\rangle$ where W is a non-empty set of worlds, N is a set of normal worlds ($N \subseteq W$), U is a non-empty scope of individuals, $\Psi(H)$ is a non-empty scope of the possible world and I is a function of interpretation.

1. If P is a predicate expression of the category s/n, then $I(P)$ is an object of the type $(2^U)^W$.

2. If $\mathbf{Q}$ is a predicate of the category $s//n$ then $I(\mathbf{Q})$ is an object of the type $(2^{(U^W)})^W$ (similarly for n-placed predicates).

3. An object of the type $(2^{(U^W)})^W$ corresponds to an intensional operator of the category $s//s$.

The essential point here is a *new way of combination of intensional* functors with their arguments (*another logical structure* of intensional contexts). Syntactically two ways of the combination can be presented: $P(a)$ if P is an extensional sign and $\mathbf{Q}[a]$ if $\mathbf{Q}$ is an intensional functor. In these cases methods of calculation of extensions and intensions on semantical level are essentially different. For extensional contexts we have:

(I) the way to determine an intension is $(A^B)^W \otimes B^W \Rightarrow A^W$; so for example $(2^U)^W \otimes U^K \Rightarrow 2^W$ is intension of $P(a)$; (I') $(A^B) \otimes B \Rightarrow A$ — the way to determine extension.

The scheme for intensional contexts is different:

(II) $(A^{B^W})^W \otimes B^W \Rightarrow A^W$; so $(2^{U^W})^W \otimes U^W \Rightarrow 2^W$ is intension of $\mathbf{Q}[a]$; (II') $(A^{B^W})^W \otimes B^W \Rightarrow A^W$ is the way to determine an extension. So the extension of a complex expression $A[B]$ depends on the intension of the argument expression B.

In accordance with two operations of application of functors to their argument two operations of abstraction are used: λxA determines a class of individuals which satisfy the condition A, and δxA determines a class of individual concepts. Accordingly, different universal quantifiers are introduced [3, pp. 241–243].

By means of simultaneous induction we introduce the concepts of intension of formula and intension of individual expression with respect to function φ for evaluation of free individual variables — $Int(A,\varphi)$ and the concept of extension in the world — $Ext_H(A,\varphi)$.

1. $Int(x,\varphi) = \varphi(x)$;
2. $Ext_H(x,\varphi) = \varphi(x)(H)$;
3. $Ext_H(R,\varphi) = I_H(R)$;
4. $Ext_H(R(\ldots,\lfloor x\rfloor,\ldots x_j,\ldots),\varphi) = t \Leftrightarrow$
   $\langle\ldots,\varphi(x),\ldots\varphi(x_j)(H)\ldots\rangle \in Ext_H(R,\varphi)$;
5. If A is a formula, then $Int(A,\varphi) = \{H \mid Ext_H(A,\varphi) = t;\}$ in other form $Ext_H(A,\varphi) = t \Leftrightarrow H \in Int(A,\varphi)$;
6. $Int(A\,\&\,B,\varphi) = Int(A,\varphi) I Int(B,\varphi)$;
7. $Int(\neg A,\varphi) = W - Int(A,\varphi)$;
8. $Int(\Box_i A,\varphi) = \{H \mid Int_H(A,\varphi) \in \Theta_i(H)\}$;
9. $Int(x = y,\varphi) = \{H \mid \varphi(x)(H) = \varphi(y)(H)\}$;
10. $Ext_H(\lambda xA,\varphi) = \{m \in U \mid \forall\varphi'(m = \varphi'(x)(H)\Lambda\varphi' =_{Hx} \varphi \Rightarrow$
    $H \in Int(A,\varphi'))\}$;
11. $Ext_H(\delta xA,\varphi) = \{w \in U^W \mid \forall\varphi'(w = \varphi'(x)(H)\Lambda\varphi' =_x \varphi \Rightarrow$
    $H \in Int(A,\varphi'))\}$;
12. $Ext_H(\Lambda(\lambda xA),\varphi) = t \Leftrightarrow U \in Ext_H(\lambda xA,\varphi) \Leftrightarrow$
    $\forall\varphi'(\varphi' =_{Hx} \varphi \Rightarrow H \in Int(A,\varphi'))$;
13. $Ext_H(\Lambda.(\lambda xA),\varphi) = t \Leftrightarrow U_H \subseteq Ext_H(\lambda xA,\varphi)$;
14. $Ext_H(\Lambda(\delta xA),\varphi) = t \Leftrightarrow U^W \subseteq Ext_H(\delta xA,\varphi)$;

15. $Ext_H((\lambda x A)(y), \varphi) = t \Leftrightarrow Ext_H(y, \varphi) \in Ext_H(\lambda x A, \varphi)$;

16. $Ext_H((\delta x A)[y], \varphi) = t \Leftrightarrow Int_H(y, \varphi) \in Ext_H(\delta x A, \varphi)$;

The introduced notions of extension and intension correspond with two methods of applying functors to argument.

The quantifiers are introduced as:

$$\Lambda x A \rightleftharpoons \Lambda(\lambda x A),$$
$$\forall x A \rightleftharpoons \forall(\delta x A),$$
$$\Lambda . x A \rightleftharpoons \Lambda .(\lambda x A).$$

According to Montague, an intensional logic can be at least second order one. Acceptance of two methods of application of functors to their arguments and two operations of abstraction respectively allows to introduce intensional predicates without any intensional operators and to construct the semantics for first-order intensional systems.

The principle of substitution of equals in the form $a = b \supset A(a) \equiv A(b)$ holds in IPL but the principle $a = b \supset A(a) \equiv A(b)$ does not hold.

Proposed approach discovers peculiarities of semantics of intensional contexts and explains why the principle of mutual replacement is violated. It gives the key for comprehension Kripke's puzzle of belief contexts [1, 2].

The peculiarities of IPL: (1) any expression including intensional predicates and operators has an intension as well as an extension; (2) intensional contexts differ from extensional by ascription of specific values to intensional predicates (operators) and, what is more important, by a way of their combination with arguments; (3) an intension of any complex extensional expression is a function of intensions of its compounds; (4) an extension of any complex intensional expression is a function of functor's extension and intensions of its arguments; (5) unlike Montague's method, this approach allows to construct an intensional logic as first-order system.

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A case for satisfaction classes: model theoretic vs axiomatic approaches to the notion of truth

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abstract. One of the basic question we can ask about truth in a formal setting is what, if anything, we gain when we have a truth predicate at disposal. For example, does the expressive power of a language change or does the proof strength of a theory increase?

Satisfaction classes are often described as complicated model theoretic constructions unable to give useful information toward the notion of truth from a general point of view. Their import is narrowed to a dimension of pure technical utility and curiosity. Here I offer an application of satisfaction classes in order to show that they can have a relevant role in confronting proof theoretical equivalent theories of truth.

Keywords: truth, satisfaction classes, axiomatic theories of truth, expansions, conservativity

1 Tarskian truth

The (broadly) Tarskian theory of truth has a prominent role in the field of formal truth theories, and it is the forced starting point for any further reflection toward the notion of arithmetical truth. Sometimes, such a theory is also called '*there is a full (not inductive) satisfaction class*' or, shortly, $PA(S)-$. It consists, apart from the axioms of the base theory PA (Peano Arithmetic in its usual first order formulation), of the truth-compositional axioms inspired by the familiar Tarskian definition of truth. In other words $PA(S)-$ is the theory in the language $LTr := L_{PA} \cup \{Tr\}$, yielded by the union of the axioms of PA in L_{PA} (which means that we do <u>not</u> have full induction) with the truth principles:

1. $\forall\varphi(Atomic(\varphi) \rightarrow Tr(\varphi) \leftrightarrow (Tr^{*}(\varphi));$

2. $\forall\varphi(Tr(\neg\varphi)) \leftrightarrow (\neg Tr(\varphi));$

3. $\forall\varphi\forall\psi(Tr(\varphi \,\&\, \psi)) \leftrightarrow (Tr(\varphi) \,\&\, Tr(\psi));$

4. $\forall\varphi\forall i((Tr(\forall vi(\varphi))) \leftrightarrow \forall t Tr(\varphi(t/vi)))$[1].

This is the usual way of writing down the axioms and, though comfortable, it is, strictly speaking, incorrect. In fact, a lot of coding apparatus has been suppressed to achieve a greater readability. To be rigorous we should write axiom 2, for example, like this:

$$\forall x\forall y\{Sent(x) \,\&\, Sent(y) \,\&\, Neg(y,x) \rightarrow [Tr(y) \leftrightarrow \neg Tr(x)]\}.$$

Here I shall persist with the most perspicuous presentation, but keep in mind the right form. The name $PA(S)-$ should then be explained, because it carries important information.

When studying truth theories, it is often said that a background theory of syntax is needed. Without it, formulating axioms for a truth predicate and working out simple operations is impossible. We need to ascribe truth to so called 'truth-bearers', and a theory of syntax is intended to give us basic information about how these entities behave. One would expect a theory of syntax to consist of principles about linguistic expressions and this was exactly the case in the original work of Tarski. However, explicit formal theories of syntax, in the style of concatenation theories i.e., are not much widespread among truth-theorists. The reason is that, after Gödel, we know that a very good deal of syntax can be developed in PA (as in even weaker arithmetical theories) and, accordingly, we can correlate natural numbers and symbols of the language of PA. There are many ways to think of this correspondence between strings of symbols and numbers, but one often adopted is the easiest one: strings of symbols are identified with corresponding numbers.

[1] I am here assuming a substitutional interpretation of quantifiers.

2 Satisfaction classes and recursive saturation[2]

2.1 Non Standard Sentences and Satisfaction Classes

Thanks to Gödel's arithmetization of syntax we can use formulas of the language of PA, L_{PA}, in order to talk about the syntax of this very language. In particular we find a correspondence between the set of sentences in L_{PA} and the elements of the domain of N, the *standard* model of arithmetic, whose domain contains all and only *standard* natural numbers.

One of the immediate consequences of compactness is that PA has, beside the standard model N, also different models, non isomorphic to N; it has *non standard* models[3]. Let M be one of these non standard models, what would happen if we were to use M instead of N as a base for arithmetization? What would happen if we coded expressions of our language using not standard elements in the domain of N but those that are in the domain of M? What would happen if we also used non standard numbers?

The first consequence would be the existence, beside *standard sentences* (those sentences coded by standard numbers), of new strange *non standard sentences*, coded by non standard elements in M[4]. In fact, among the many syntactical properties that can be represented in PA we can obviously define that of being a sentence of the language of PA, since there is a formula '$Sent(n)$' which is true of n if and only if n is a code of a sentence of L_{PA}. Until we consider the standard model of PA, as is natural doing, this works as expected. However, in non-standard models, the formula '$Sent(x)$' is going to be satisfied by non-standard numbers too. The reason is the *Overspill Principle*. According to it, if a formula is such that infinitely many *standard* numbers satisfy it, then — when we have a non-standard model — some *non-standard* number will satisfy it too. Clearly the formula '$Sent(x)$' is satisfied by infinitely many

[2]The literature on satisfaction classes and recursive saturation is highly technical and difficult. We make general reference to [6], [2], [9], [8]. Personally, I have to thank Fredrik Engström for his patience in explaining me the quibbles of satisfaction classes. I certainly owe what I have understood (if any) to him and his long mails.

[3]For a good brief introduction to non standard models see [1, Ch. 25]

[4]The fundamental work is [13].

standard numbers, so when $M \models PA$ is non standard, we have that $M \models Sent(b)$ for some $b \in M$, and b non standard (similar for the syntactical notion of Term, Formula and so on). The existence of non-standard numbers that, according to the model, code *sentences*, drags us towards the realm of non-standard sentences. Very roughly, non standard sentences are sentences with a *non-standard structure*.

Non-standard sentences can be identified with non-standard elements that the model M 'thinks' to be actual sentences (those non-standard numbers that code sentences in the sense of M). It is not easy to give a clear description of what non standard sentences are. I propose just an example. Consider the sentence in $L_{PA}(\neg 0 = 0)$. This is a case of a standard sentence that N (and then M) recognizes to be a sentence, and such that it can be identified with its standard natural number of Gödel. We have a similar example with $(\neg 0 = 0)\&(\neg 0 = 0)$ and $(\neg 0 = 0)\&(\neg 0 = 0)\&(\neg 0 = 0)$, where the number of conjuncts is a standard natural number, (2 and 3 respectively). If the number of conjuncts is a non-standard number, for instance $(\neg 0 = 0)\&(\neg 0 = 0)\&\ldots\&(\neg 0 = 0)$ (where the dots '$\ldots$' stand for a-many repetitions of the sentence $(\neg 0 = 0)$, and a is a non-standard number) we do not deal with a standard sentence anymore (N cannot recognize it as a sentence), we have obtained a non-standard sentence and M (if it contains a) can recognize it to be a sentence.

Regarding non-standard sentences a natural question is whether and how they are true. We know that a truth predicate, 'Tr' such that $N \models Tr([\varphi]) \leftrightarrow \varphi$ for every sentence φ (where $[\varphi]$ is the code of φ), is not definable in L_{PA} (for Tarski's undefinability theorem). This is the reason why we had to add such a new predicate together with axioms governing it, obtaining the truth theory $PA(S)-$. The same claim holds for non-standard sentences, since, a fortiori, a predicate 'Σ' such that $M \models \Sigma([\varphi]) \leftrightarrow \varphi$, for every sentence φ in the sense of M (standard and non-standard) cannot be defined in M. What should such a predicate Σ be like? First of all it should agree, at least, with the truth predicate for standard sentences, namely it should respect Tarskian clauses. Here is where the notion of *satisfaction class* must be introduced.

Once we have added the truth predicate to PA and we have obtained our truth theory $PA(S)-$, we need to find an extension S in M for such a new predicate. Namely, given a model $M \models PA, S$ is supposed to be the set of numbers satifying the axioms of $PA(S)-$, $M, S \models PA(S)-$. When, in a model $M \models PA$, such a set S is available, we say that S is a *satisfaction class* for M. This explains the name '*PA plus there is a full (not inductive) satisfaction class'*. A satisfaction class[5] S over a model M, then, is a set S of elements in M, where any element b in S is such that $M \models Sent(b)$ and b satisfies the Tarskian clauses for truth, namely the axioms of $PA(S)-$. In other words, S is a suitable extension for the truth predicate, as governed by Tarskian axioms, possibly including (codes of) non-standard sentences.

Definition 1.[6] If M is a (non-standard) model of PA, a subset S of M is a *satisfaction class* if and only if: $M, S \models PA(S)-$.

Satisfaction classes can be classified further as follows.

Definition 2. A satisfaction class S on M is *full* if, for every $M \models Sent([\varphi])$, we have that $[\varphi] \in S$ or $[\neg\varphi] \in S$.

Definition 3. A satisfaction class S on M is *partial* if and only if there is α belonging to $M \setminus N$ such that if $M \models Sent[\varphi]$ and $[\varphi] < \alpha$, then $[\varphi] \in S$ or $[\neg\varphi] \in S$.

The idea here is just that a satisfaction class is *full* if, for every formula φ, S contains either φ or its negation and, if the satisfaction class is *partial* this is true only for those sentences coded by a (non-standard) number less than α. Since standard sentences are coded by standard natural numbers and every standard natural number is less than every non-standard natural number, it follows that every satisfaction class (full or partial) behaves in the same way (they are full) with respect to standard sentences. It is important to notice that a satisfaction class, even if it is a partial one, has to decide some non-standard sentence, otherwise we have not a satisfaction class at all[7].

[5] Though it is called 'class' it is a set.
[6] See [8] and [11].
[7] I owe this important remark to Fredrik Engström.

Indeed, if M is non standard we always have some non standard sentence into the extension of 'Tr'. Since every axiom of $PA(S)-$ is subjected to a clause stating that the truth predicate applies to elements satisfying the formula '$Sent(x)$', when we have a non-standard model, non-standard numbers can well enter into the range of the truth predicate. Actually this is not only possible but mandatory. In fact, $PA(S)-$ proves $\forall\varphi\{Sent(\varphi) \rightarrow [(Tr([\varphi]) \vee Tr([\neg\varphi]))]\}$, thus, for every φ such that $M \models Sent([\varphi])$ either φ or $\neg\varphi$ must be in the extension of 'Tr', even if φ is non-standard.

DEFINITION 4. A satisfaction class is *inductive* if and only if the expanded structure (M, S) satisfies all the induction axioms for every formula in the language $L_S = L_{PA} \cup \{S\}$ (Where the new symbol 'S' is governed by axioms stating that S is a satisfaction class. In our cases 'S' is substituted with 'Tr').

If this is the case, we have '*PA plus there is a full inductive satisfaction class*', turning from $PA(S)-$ to $PA(S)$. These two theories have very different features and strength, but I shall mostly consider $PA(S)-$ only.

Crossing these definitions we can get other classifications by distinguishing, with respect to full and partial satisfaction classes, those satisfaction classes that are inductive and those that are not.

We saw that, as far as standard sentences are concerned, a satisfaction class agrees with the traditional characterization of Tarskian truth. The surprising news is that classical compositional axioms are not enough to shape the truth of non-standard sentences. In other words, in order to characterize the truth (or falsity) of non-standard sentences we need some other tool than just compositional clauses in their standard formulation. If we only stick with Tarskian axioms, then we are free of constructing many different satisfaction classes such that they will agree on standard sentences but will disagree on many non-standard sentences. As a result, if a model M admits a satisfaction class, then it admits *many* satisfaction classes. Indeed, if S is a full satisfaction class for a countable M, then there are continuum many non isomorphic expansions (M, T) which are all elementarily equivalent to (M, S).

This richness should be contrasted with this other fact: though a model can have many different satisfaction classes, not every model of PA can have one. Non *recursively saturated models*, in fact, cannot have any satisfaction class.

2.2 Recursive saturation

In order to explain the notion of *recursive saturation* we need some minimal elementary information[8].

DEFINITION 5. If B is a theory, a *type* over B is:

(i) a set $P(\underline{x})$ of formulas containing a finite number of free variables $\underline{x}$ ('$\underline{x}$'stands for a sequence of variables).

(ii) $P(x)$ is such that $B \cup \{\varphi(\underline{c}) \mid \varphi(\underline{x}) \in P(\underline{x})\}$ is consistent. (Where '$\underline{c}$' stands for a sequence of — possibly new — individual constants).

DEFINITION 6. A type $P(x)$ is *complete* if and only if $T \cup P(x)$ is a syntactical complete theory (that is for every $\varphi(\underline{x}), T \cup P(\underline{x}) \vdash \varphi(\underline{x})$ or $T \cup P(\underline{x}) \vdash \neg\varphi(\underline{x})$).

DEFINITION 7. A type $P(\underline{x})$ is *principal* if and only if there is a single formula $\psi(\underline{x})$ such that $T \vdash \forall x(\psi(\underline{x}) \rightarrow \varphi(\underline{x}))$, for every $\varphi(\underline{x}) \in P(\underline{x})$.

DEFINITION 8. If $M \models B$, a type $P(\underline{x})$ is *realized* in M if and only if there is $\underline{a} \in M$, such that $M \models, \varphi(\underline{a})$, for every $\varphi(\underline{x}) \in P(\underline{x})$. Otherwise M *omits* the type $P(\underline{x})$.

For completeness theorem, if $P(\underline{x})$ is a type over a theory B, then B has a model that realizes $P(\underline{x})$. Similarly, if $P'(\underline{x}), P''(\underline{x})\ldots$ are types over the theory $Th(M)$ of a model M (that is the set of all the sentences φ such that $M \models \varphi$), then there is an elementary extension M' of M that realizes every type $P(\underline{x})$.

DEFINITION 9. A type $P(\underline{x})$ is *recursive* if the set $\{\varphi(\underline{x}) \mid \varphi(\underline{x}) \in P(\underline{x})\}$ is recursive. (Notice that what is recursive is the set of formulas, not the formulas, which can have whatever complexity.)

[8] It is possible to give the following definitions also in model-theoretic terms instead of talking of theories. I use the proof theoretic definition in order to stress the relation of this notion with the notion of truth as axiomatized by $PA(S)-$.

DEFINITION 10. A model M is *recursively saturated* if and only if every recursive type over $Th(M)$ is realized in M.

A recursively saturated model can be thought as a *big* and *homogeneous* model. A non-recursively saturated model O is a model where at least one recursive type (a recursive set $P(\underline{x})$ of formulas) is not realized in O. This means that there are formulas $\varphi(\underline{x}) \in P(\underline{x})$, for which elements $\underline{a}$ in O such that $O \models \varphi(\underline{a})$ are not available. This can happen, for example, when the model is not *homogeneous* or it is not *big* enough. To make it such, we need to extend O to O' adding new elements $\underline{a}$ with the desired features.

The fundamental fact now is that if a non-standard model admits a satisfaction class, then such a model needs to be big and homogeneous in this sense: it must be recursively saturated. This is the sense of Lachlan's theorem:

THEOREM 1 (LACHLAN'S THEOREM[9]). *If M is a non-standard model of PA with a full not inductive satisfaction class, then M is recursively saturated.*

It is possible to get a similar result also for partial satisfaction classes too:

THEOREM 2.[10] *If M is a non-standard model of PA with a partial satisfaction class, then M is recursively saturated.*

We can sum up the story stating that if M is non-standard and it has a satisfaction class (it does not matter whether full or partial, or whether it is inductive or not), then M must be recursively saturated. Recursive saturation is a necessary condition for a non-standard model to have a satisfaction class[11].

However, recursive saturation is not a sufficient condition to guarantee the possibility of a satisfaction class: in fact there exist non countable recursively saturated models without a full satisfaction class or an inductive satisfaction class[12]. Recursive saturation of a

[9] [12]

[10] See [6, Theorem 15.5 and proposition 15.4].

[11] Notice that this is not true for the standard model N. N is not recursively saturated but it does admit a 'satisfaction class'.

[12] See [5].

non-standard model is a sufficient condition to have a satisfaction class only together with countability:

THEOREM 3. *If M is a countable recursively saturated model of PA, then M admits a satisfaction class.*

Notice that this does not mean that every countable recursively saturated model of PA admits a satisfaction class whatsoever: for instance it is not enough to have a full *inductive* satisfaction class. In fact, $PA(S)$ is famously able to prove the consistency of PA, so that just models where $Con(PA)$ holds can be expanded to models of $PA(S)$, and some recursively saturated models are excluded.

It is very important for us to stress that **there are non-standard models of PA such that they are not recursively saturated**. Therefore, there are non-standard models of PA that do not admit a satisfaction class.

3 Satisfaction classes and axiomatic truth theories

With such results available we can draw some important consequences. The first observation is rather natural and concerns the relation between satisfaction classes and axiomatic theories of truth. The notion of satisfaction class has been constructed with the purpose of characterizing the set of all arithmetical truths in a certain model from a model-theoretic point of view, while an axiomatic setting tries to characterize the behaviour of the truth predicate. It is clear that such approaches can be considered, in a certain measure, as two different ways of working at the same problem. We can then expect an axiomatic theory of truth to give an axiomatization of a predicate defining a satisfaction class[13].

The second, more important observation, is that a model-theoretic approach to truth can enlighten aspects that a pure proof theoretic investigation is not able to show. $PA(S)-$ gives us the clearest example of such a situation. In fact, from a proof theoretic

[13]In our definition of satisfaction class we used a relation symbol to talk about the satisfaction of a formula by a sequence of objects, while in the axiomatic theories we are using a one-place truth predicate. This difference, however, has not deep effects, at least with regard to our problems. It would have been possible, for example, to define a satisfaction class avoiding the notion of satisfaction (as in [2]).

point of view, $PA(S)-$ is a *conservative* extension of PA. So with the addition of the truth axioms in $PA(S)-$, apparently, we do not get any new arithmetical information, and, viewed from PA, this enrichment looks rather useless. We might be tempted to say that PA and $PA(S)-$ have the same arithmetical content. This, however, would be a mistake as the application of satisfaction classes will show.

To better explain the point, consider the theory $ACA-$[14], which is the axiomatic theory for second order arithmetic (*Arithmetic* with *Comprehension Axioms*), yielded adding to PA axioms for second order comprehension preventing full induction. More precisely: $ACA-$ is formulated in the second order language $L2$ of second order arithmetic and is given by adding to PA the axioms

$$\exists X \forall y (y \in X \leftrightarrow \varphi)$$

where φ is a formula of $L2$ without any second order quantifier or X. Notice that in $ACA-$ there is arithmetical induction only.

As is well known $ACA-$ is conservative over PA. Moreover $ACA-$ can define a truth predicate for PA. Namely, in $ACA-$ it can be defined a formula $\tau(x)$ such that $ACA- \vdash, \tau([\varphi]) \leftrightarrow \varphi$. (Though a truth predicate respecting Tarskian clauses cannot be defined in $ACA-$.) Again a natural expectation might be that $ACA-$ and $PA(S)-$ should be two arithmetically equivalent theories, sharing the same *arithmetical content*. This is the natural conclusion given the fact that both additions are conservative over PA, so that they prove the same arithmetical theorems. Indeed, if we have full induction, turning to ACA and $PA(S)$[15], we obtain an interdefinability result. As is well known, ACA can define the truth predicate of $PA(S)$, and $PA(S)$ can define membership of ACA. Thus, the moment we drop full induction, even if we lose this pleasant interdefinability, we might expect the equivalence to keep holding. As a matter of fact, however, these two theories are not arithmetically equivalent and the proposed proof theoretic analysis did not give us all relevant information. If we turn to a model

[14] See [4].

[15] These theories are exactly the same as $ACA-$ and $PA(S)-$ except from the fact that induction of PA is now extended to these new languages.

theoretic approach, instead, we see clearly that they have a deeply different arithmetical impact.

$ACA-$ is conservative over PA and each model of PA can be expanded to a model of $ACA-$. This is actually how the conservativity of $ACA-$ is usually proved. So, in this sense, $ACA-$ can be thought to really have the same arithmetical content of PA. $ACA-$ keeps characterizing all the models of PA. Hence, if we see PA as a way to talk about a large class of (arithmetical) structures, $ACA-$ can be seen as another way to talk of the very same class. This is a strengthening of the conservativity result: not only do not we get new arithmetical theorems, we do not restrict the number of possible models either.

From the supposed equivalence between $ACA-$ and $PA(S)-$ one would expect a similar situation to keep holding even for $PA(S)-$, which, in fact, is conservative over PA too. But, as we know, thing are very different. Indeed, a non-standard model M of PA is a model of $PA(S)-$ if and only if M admits a satisfaction class. Unfortunately, not every model of PA does. If a non-standard model of PA is not recursively saturated, then it is no use trying to expand it to $PA(S)-$. The reason is just Lachlan's theorem.

There is a kind of asymmetry between $ACA-$ and $PA(S)-$ then. Both are conservative over PA, but they affects the models of PA in very different ways. $PA(S)-$ cannot be taken to be another way of characterizing the same class of structures since it is able to exclude some of the models of PA. Thus, claiming that $PA(S)-$ has the same arithmetical content of PA is not plainly correct. It has extra-content, for it requires the models to be in a certain precise way: they must be recursively saturated.

4 A philosophical application

A philosophical insight into the notion of truth is deeply related to similar issues. Consider, for example, a deflationist approach to truth, which is probably the most debated proposal nowadays. Deflationism rejects the traditional philosophical explanation of the concept. In particular, according to it, truth is claimed not to be a very deep notion, and the property it stands for, if any, not to have any metaphysical structure or weight. Opposite to traditional approaches, like correspondence theories, truth is claimed to lack any robust ontological import: truth is an unsubstantial property.

Such an unsubstantiality has been explained by exploiting the fact that, if truth is characterized using some very simple axioms, exactly like typical Tarskian ones, we can easily get conservative extensions of PA in a number of cases. Since, then, conservative theories are thought to be innocent additions, by using the notion of conservativity it is possible to make sense of the metaphysical innocence of truth. If a theory T is conservative over a theory B, T does not prove anything new regarding what B is about, so that, it has been suggested, viewed from B, the addition of T is redundant. If T is our theory of truth, then, truth is harmless in this sense, as deflationists argue. Consequently, the unsubstantiality of truth can be identified with the conservativity of its theory. If a truth theory is conservative, it will not prove anything apart from semantical claims, and it will not be capable of concrete (read extra semantical) power. The conservativity of a theory of truth can be taken to be evidence for the evanescent nature of the property of truth.

Things, however, are not so easy. In fact, as our previous analysis showed, there is much more content in a theory than that that can be enlightened by merely proof theoretical means. Possibly, a theory T can be conservative over B, while not every model of B be expandable to a model of $T \cup B$, so that T can still have some impact over what B is about. Conservative theories can as well affect the content of the theory they are added to. A truth theory like $PA(S)-$ is conservative over PA, nevertheless it makes a difference. $PA(S)-$ has a rich and interesting arithmetical content which is at least as rich as the notion of recursive saturation, a very pivotal tool in model theory. It follows that conservativity is not a suitable candidate to analyse the alleged unsubstantiality and innocence of truth; exactly as a mere proof theoretic approach is not enough to make us able to draw all the relevant consequences.

5 Conclusion

When we take an axiomatic approach to truth, the customary way of proceeding is that of devising a suitable set of axioms governing the truth predicate and then studying the logical properties of such an axiomatization. In particular, a certain axiomatic theory of truth is evaluated with respect to its impact over an arithmetical theory like PA. We check whether it is conservative, whether it is able to prove new arithmetical theorems or whether it is able to prove the consistency of PA and so on. Once we have that, it is interesting to assess what the proof strength of our theory is with respect to other proposals on the market. We try to put our new axiomatization in the right place of the hierarchy.

The reflections I proposed are mainly motivational ones. They are intended to show that a merely proof theoretical evaluations of theories, and of truth theories in particular, are not, sometimes, fine grained enough. There might be more differences than those emerging in proof theoretical terms: two theories with the same arithmetical proof strength can have very different impacts over the model of PA, and exhibit a different arithmetical content. This is absolutely relevant if we are interested in identifying the correct power of truth theories. But this is also important from a general philosophical point of view, since, if this point is neglected, metaphysical misconceptions loom.

The notion of truth is traditionally tackled, into a formal framework, by building either axiomatic systems or semantical interpretations. Where, in the latter case, one proposes a model in which the truth predicate is interpreted in some nice way. Semantical and axiomatic approaches do not interact many often, so that the connection between them is rather weak. This is certainly unpleasant from a general point of view: a model theoretical approach can be crucial to enlighten important aspects of axiomatic theories and can offer an important contribution to their evaluation. The considerations above showed a case of this possibility and can be generally considered as a first step toward a closer collaboration between an axiomatic and a semantical approach to the notion of truth.

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Proto-Entailment in RS logic

Dmitry V. Zaitsev[1]

A jump into abstraction – performed in
universal algebra and universal logic –
allows space for monsters.
J-Y. Bézeau

abstract. In this paper I propose a formalization of proto-entailment relation introduced by V. Shalak by means of RS logic. The first section clarifies the idea and formal developments of RS logic, which is the logic of Rational Subject. In the second section I will very briefly introduce the conception of proto-entailment as it was promoted in Shalak's writings. The third section contains the formal account for proto-entailment and axiotimatization of resulting logic.

Keywords: proto-entailment, logic of rational subject, generalized truth-values

1 Logic of Rational Subject

The abbreviation *RS-logic* expansion is [logic] 'of Rational Subject', that is a four-valued propositional logic, whose values are two-component entities composed of logical and epistemological constituents. First the idea of such a logic emerged in the course of working on the project of generalized classical truth values [4]. We elaborated an idea of distinguishing between ontological and epistemological aspects of classical truth values. In so doing, we came across two unary twin connectives that deal only with either ontological or epistemological component of generalized classical truth value, leaving the other untouched. That is why these connectives were labeled as semi-classical negations. I turned onto whether there

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is any logic wherein any of our semi-negations is treated as full-scale one. This stream of thought led me to the logic of Rational Subject.

Imagine a rational subject who knows the laws of classical logic. It means that when it is necessary to calculate the value of any compound formula from the values of its constituent formulas our subject performs computations guided by a knowledge of classical truth-assignments. It is evident that proceeding along these lines he (or she) sometimes can figure out the value of a formula, and, thus, *knows* its value, and sometimes the information in hand is not enough to fix the value of the formula, and when this occurs, our subject *does not know* the value. Hence, besides two 'logical' (or ontological) values Truth and Falsity one must take into account two extra 'epistemic' values characterizing the state of rational subject's knowledge. Let Truth and Falsity as usual be denoted by 't' and 'f', while for 'knows' and 'does not know' we select '1' and '0' correspondingly. Then we have just four values being two-component entities composed of logical and epistemological constituents that can be treated as pairs or as sets:

T1 $\langle t, 1\rangle$ $\{t, 1\}$;
T0 $\langle t, 0\rangle$ $\{t\}$;
F0 $\langle f, 0\rangle$ $\{f\}$;
F1 $\langle f, 1\rangle$ $\{f, 1\}$.

Consider the clauses for negation and conjunction to clarify the way rational subject works. If rational subject knows that an arbitrary formula is true, he knows that its negation is false, and vice versa. In the meantime, if you do not know the value of a formula, you do not know the value of its negation. The resulting truth-table goes as follows in Figure 1.

A	$\neg A$
T1	**F1**
T0	**F0**
F1	**T1**
F0	**T0**

Figure 1. Table for 'rational' semi-classical negation

Due to its classical nature conjunction is true if and only if both conjuncted formulas are true, that determines the first component of values-as-pairs (occurrence of element 't' in a value-as-set). The second epistemic component of a value can be calculated on the basis of the following reflections: one knows that conjunction is true if and only if one knows that both conjuncts are true, and one knows that conjunction is false if and only if one knows that at least one of conjuncts is false. Summing up these considerations we receive the truth-table for conjunction depicted in Figure 2. The analogous

$\wedge$	**T1**	**T0**	**F0**	**F1**
T1	**T1**	**T0**	**F0**	**F1**
T0	**T0**	**T0**	**F0**	**F1**
F0	**F0**	**F0**	**F0**	**F1**
F1	**F1**	**F1**	**F1**	**F1**

Figure 2. Table for 'rational' conjunction

argument provided for disjunction clause makes it possible to consider the structure of generalized values as a four-elements lattice with linear ordering depicted in Figure 3.

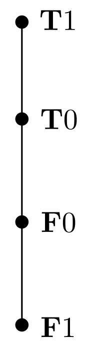

Figure 3. The 'rational' lattice

Now define a valuation function v as a map from the set of propositional variables to the set $V = \{T1, T0, F0, F1\}$, and in a straight forward way extend it to arbitrary formula A given correspondence between lattice meet (join) and conjunction (disjunction). Thus, we have a four-valued valuational system, which allows to define different consequence relations on it.

For a period I had been zeroing in on the different problems put RS logic aside. However it was my student Yekaterina Kubyshkina who in her graduation thesis and relevant publications [3] examined some consequence systems, which axiomatize RS logics with different consequence relations. In particular, she introduced three consequence relations: $\forall A, B$

- $A \models_{RM} B \Leftrightarrow \forall v(v(A) \in D \Rightarrow v(A) \in D)$,
  where $D = \{T1\}$ — T1-preserving consequence;
- $A \models_{TV} B \Leftrightarrow \forall v(v(A) \in D \Rightarrow v(A) \in D)$,
  where $D = \{T1, T0\}$ — truth-preserving consequence;
- $A \models_{KL} B \Leftrightarrow \forall v(v(A) \leq v(A))$,
  where $\leq$ is the 'rational' order — comparative consequence.

She has proved that corresponding semantical logics can be presented as consequence systems: RS_{RM} (RS with $\models_{RM}$) is axiomatized by the first-degree fragment of RM; RS_{TV} (RS with $\models_{TV}$) is axiomatized by classical consequence system; RS_{KL} (RS with $\models_{KL}$) is axiomatized by Kleene strong logic.

Immediately, a string of questions arises. And among them the following directly pertains to the topic of this paper: what is the consequence system axiomatizing a 1-preserving entailment? To answer this question we first turn to Shalak's idea of proto-entailment.

2 Proto-Entailment

Modern Russian logician Vladimir Shalak set forth an idea of proto-entailment proceeding from radically different intuitive premisses. The title of his doctoral dissertation is 'Proto-Logic: new insight into the nature of logicality' and he sees his primary objective in clarifying the very concept of logic. His approach is very close to the so called project of universal logic, which pretends to be a general theory of logics. Shalak himself highlights the cognation of his theory with the ideas of J-Y. Bésiau [1, 2]. J-Y. Bésiau interprets universal logic by analogy with universal algebra: the latter is an abstract set of formulas together with equally abstract consequence relation subject for no specific restrictions.

Meanwhile, it is worth noting that Shalak himself in English-language abstract to his papers (published in Russian) uses the term 'consequence relation'. However he oftentimes emphasizes that this consequence relation is free from well-known paradoxes, and hence makes a good name of (proto-) entailment for it.

However even such an abstract relation needs a precise definition. This is a fragment from his paper devoted to an alternative definition of consequence relation that helps to grasp the underlying informal intuition.

> In classical logic, truth of premises is a sufficient condition of verity of the conclusion. However, that is too stronghold limiting a requirement. A laxer claim might be to have valid ways of reasoning simply not lead us to erroneous conclusion or fallacies...
>
> In other words, the form of the argument is valid if the knowledge of its premises' truth-value is a sufficient condition of the awareness of its conclusion's truth-value... [5, p. 283].

To understand why and how Shalak proceeded from this informal motivation to the axiomatically presented proto-Boolean logic, one should take into account the other, maybe even most important for him, idea that constitutes his conception of modern logic. One of his fundamental presumptions is that the radical turn from subject-predicate paradigm in logic to relational one neither was necessarily determined nor offered any advantage in a formal language expressive power. In fact, he suspects that as a result of such a paradigm shift our world-view has been distorted. It would be more natural and convenient to develop symbolic logic on the ontological basis of (monadic) properties and functions rather than on the ground of relational structures. This presuppositions have strongly influenced on his further formal explication of consequence relation, which he defines functionally as follows.

> This [the above consideration] gives rise to the following definition of the entailment relation:

> The set of formulas $\Sigma = \{B_1, \dots, B_k\}$ entails formula A, iff there exists function f which allows calculation of the truth-value of A given truth-values of the formulas of the set Σ. [5, p. 283].

Quite predictably this function f turns to be Boolean one that provides extremely plain axiomatization of proto-Boolean logic as a consequence system **ACL** (that is *alternative consequence logic* or *alternative to classical logic* , as may well be imagined). There are just three axiom schemes and two rules:

A1 $A \vee \neg A$;

A2 $\{A, B\} \vdash_{SH} A \wedge B$;

A3 $\{A\} \vdash_{SH} \neg A$;

R1 $\dfrac{\vdash_{TV} A \equiv B}{\{A\} \vdash_{SH} B}$; **R2** $\dfrac{\Gamma \vdash_{TV} A,\ \{A\} \cup \Delta \vdash_{SH} B}{\Gamma \cup \Delta \vdash_{SH} B}$,

where $\vdash_{SH}$ and $\vdash_{TV}$ stand for Shalak's proto-entailment and classical consequence relation correspondingly.

It can be easily shown that $\vdash_{TV} A \Rightarrow \vdash_{SH} A$, thereby validating in **ACL** all classically valid formulas.

In his resent writings, Shalak makes an attempt to formalize more abstract *functional* concept of proto-entailment. However, currently he has only suggestive axiomatization of corresponding consequence relation.

3 Proto-Entailment as a 1-Preserving Consequence Relation

In what follows, I will present a consequence system RS_{PE} which formalizes proto-entailment as a 1-preserving consequence relation in RS logic. In so doing, first consider semantics for RS_{PE} in more detail.

For the sake of convenience, in this section, the values of RS logic will be interpreted as sets (sf. the 1st section). A valuation function v is the map from the set of propositional variable into the four-element set of values-as-pairs, extended to compound formulas in a straightforward way as provided by the truth-tables above. Then the following proposition can be put forward.

Proposition 1.
$t \in v(\neg A) \Leftrightarrow f \in v(A) \quad f \in v(\neg A) \Leftrightarrow t \in v(A)$
$1 \in v(\neg A) \Leftrightarrow 1 \in v(A)$
$t \in v(A \wedge B) \Leftrightarrow t \in v(A)$ *and* $t \in v(A)$
$f \in v(A \wedge B) \Leftrightarrow f \in v(A)$ *or* $f \in v(B)$
$1 \in v(A \wedge B) \Leftrightarrow [1 \in v(A)$ *and* $1 \in v(B)$ *and* $t \in v(A)$ *and* $t \in v(B)]$ *or* $[1 \in v(A)$ *and* $f \in v(A)]$ *or* $[1 \in v(B)$ *and* $f \in v(B)]$.

Definition 1. For arbitrary formulas A and B of L_{RS}, $A \models_1 B \Leftrightarrow \forall v(1 \in v(A) \Rightarrow 1 \in v(B))$.

A consequence system RS_{PE} is presented as pair $(L_{RS}, \vdash)$, where $\vdash$ satisfies the following deductive postulates:

A1. $A \vdash \neg A$

A2. $\neg A \vdash A$

A3. $A \wedge (B \vee C) \vdash (A \wedge B) \vee C$

A4. $A \wedge B \vdash \neg A \vee \neg B$

A5. $\neg A \vee \neg B \vdash A \wedge B$

A6. $A \vee B \vdash \neg A \wedge \neg B$

A7. $\neg A \wedge \neg B \vdash A \vee B$

R1. $\dfrac{A \vdash B, B \vdash C}{A \vdash C}$ **R2.** $\dfrac{A \vdash B, A \vdash C}{A \vdash B \wedge C}$

R3. $\dfrac{A \vdash B, C \vdash B}{A \vee C \vdash B}$ **R4.** $\dfrac{A \vdash B, A \vdash_{TV} \neg B}{A \vdash B \wedge C}$

R5. $\dfrac{A \vdash B \wedge C, A \vdash_{TV} B \wedge C}{A \vdash B, A \vdash C}$ **R6.** $\dfrac{A \vdash B \wedge C, A \vdash_{TV} \neg B}{A \vdash B \text{ or } A \vdash C}$,

where $\vdash_{TV}$ designates classical consequence relation.

There are some interesting and helpful theorems:

t1. $A \wedge B \dashv\vdash \neg(\neg A \vee \neg B)$

t2. $A \vee B \dashv\vdash \neg(\neg A \wedge \neg B)$

t3. $A \wedge \neg A \dashv\vdash A$

t4. $C \wedge (A \wedge \neg A) \dashv\vdash C \wedge A$

t5. $A \vdash A$

The proof of soundness is mostly a routine check which can be smoothly omitted.

THEOREM 1 (COMPLETENESS). *For any A and B of L_{RS}: If $A \models_1 B \Leftrightarrow A \vdash B$.*

PROOF. Suppose $A \models_1 B$. To show that $A \vdash B$, define canonical valuation via consequence relation as follows:
$v_c(p) = T1 \Leftrightarrow A \vdash_{TV} p$ *and* $A \vdash p$;
$v_c(p) = T0 \Leftrightarrow A \vdash_{TV} p$ *and* $A \nvdash p$;
$v_c(p) = F0 \Leftrightarrow A \vdash_{TV} \neg p$ *and* $A \nvdash p$;
$v_c(p) = F1 \Leftrightarrow A \vdash_{TV} \neg p$ *and* $A \vdash p$.
To simplify the proof consider only three generalized conditions for canonical valuation so defined:

1. $t \in v_c(p) \Leftrightarrow A \vdash_{TV} p$;
2. $f \in v_c(p) \Leftrightarrow A \vdash_{TV} \neg p$;
3. $1 \in v_c(p) \Leftrightarrow A \vdash p$.

Now we need to prove that the canonical valuation for arbitrary formula B satisfies conditions 1–3.

The first point that strikes the eye is the possibility for conditions 1 and 2 to coincide. If $A \vdash_{TV} B$ and $A \vdash_{TV} \neg B$ then $A \vdash_{TV} B \wedge \neg B$. The latter means that A is of the form $F \wedge (C \wedge \neg C)$. By **t3.** and **t4.**, $F \wedge (C \wedge \neg C) \dashv\vdash F \wedge C$. Let now A^* be $F \wedge C$. As long as according to our basic assumption $A \models_1 B$, to show that $A \vdash B$ is equivalent to show that $A^* \vdash B$, reducing the case to non-contradictory one.

The proof for arbitrary formula B will be carried out by simultaneous induction on the length of a formula. Keeping in mind theorems **t1.** and **t2.** one can consider only cases with negation and conjunction.

Case ($\neg B$) and conditions 1–3 hold.
1) $t \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} f \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash_{TV} \neg B$
2) $f \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} t \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash_{TV} B$
$\Leftrightarrow_{[PC]} A \vdash_{TV} \neg\neg B$
3) $1 \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} 1 \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash B$
$\Leftrightarrow_{[A1.,A2]} A \vdash \neg B$

Case ($B \wedge C$) and conditions 1–3 hold.
1) and 2) are routine.
3) $1 \in v_c(B \wedge C) \Leftrightarrow_{[prop.1.]} [1 \in v_c(B)$ *and* $1 \in v_c(C)$ *and* $t \in v_c(B)$ *and* $t \in v_c(C)]$ *or* $[1 \in v_c(B)$ *and* $f \in v_c(B)]$
or$[1 \in v_c(C)$ *and* $f \in v_c(C)]$
$\Rightarrow$:
$1 \in v_c(B)$ *and* $1 \in v_c(C)$ *and* $t \in v_c(B)$ *and* $t \in v_c(C) \Rightarrow_{[ind.\ assum.]}$
$A \vdash B$ *and* $A \vdash C \Rightarrow_{[R.2.]} A \vdash B \wedge C$
$1 \in v_c(B)$ *and* $f \in v_c(B) \Rightarrow_{[ind.\ assum.]} A \vdash B$ *and* $A \vdash_{TV} C \Rightarrow_{[R.4.]}$
$A \vdash B \wedge C$
$1 \in v_c(B)$ *and* $f \in v_c(B)$ is analogous
$\Leftarrow$: $A \vdash B \wedge C \Rightarrow_{[R.5.]} (A \vdash B, A \vdash C, A \vdash B \wedge C)$ *or*
$(A \vdash B, A \vdash_{TV} \neg B)$ *or* $(A \vdash C, A \vdash_{TV} \neg C) \Rightarrow_{[ind.\ assum.]}$
$[1 \in v_c(B)$ *and* $1 \in v_c(C)$ *and* $t \in v_c(B)$ *and* $t \in v_c(C)]$
or $[1 \in v_c(B)$ *and* $f \in v_c(B)]$ *or* $[1 \in v_c(C)$ *and* $f \in v_c(C)] \Leftrightarrow_{[prop.1.]}$
$1 \in v_c(B \wedge C)$

Turning to completeness, $A \models_1 B$ means that $\forall v(1 \in v(A) \Rightarrow 1 \in v(B))$. And hence $1 \in v_c(A) \Rightarrow 1 \in v_c(B)$, that is $A \vdash A \Rightarrow A \vdash B$. $A \vdash A$ holds by **t5.**, and by MP, $A \vdash B$, completing the proof. □

4 Conclusion

The aim of this paper was twofold.

(1) *Formalization of proto-entailment.* What kind of (proto-)entailment, in Shalak's sense, is formalized by the system RS_{PE}? Is it Boolean proto-entailment or general functional one? The answer that suggests itself is that the 1-preserving consequence relation corresponds to Boolean proto-entailment. The main argument in favour of such a conclusion is the apparently classical justification

of the truth-conditions for compound formulas exploited in the first section. Though one can call in question this reflection by asking if any other (non-classical) interpretation of propositional connectives are entitled to exist. Maybe it would be useful to consider non-Fregean logical components of compound values? As I see it, presumably this trick won't come off. The proliferation of logical components will complicate the assignment procedure and bring about to loss of clarity in compound values interpretation. Thus, the presented formalization can pretend to be the explication of general functional proto-entailment as well.

What is worth noting is the relationship between the 1-preserving entailment and the t-preserving entailment ($\models_{TV}$): if $A \models_1 B$, then $A \models_{TV} B$ or $A \models_{TV} \neg B$ but not vice versa. Supporting my previous claim the same relation holds between $\models_1$ and $\models_{RM}$ and between $\models_1$ and $\models_{KL}$!

(2) *Prospects for RS logic.* Digressing from proto-entailment, this logic seems of certain interest in itself.

First, it opens possibilities for a wide range of different consequence relations defined for instance via transfer from truth of premises to the awareness of conclusion and so on.

Second, *RS* logic is still waiting to be supplied with appropriate implication(s). Interestingly, natural classical style implication can be easily added as an abbreviation for $\neg A \vee B$. However this simple-mind implication turns to be a Kleene's one. Regarding Łukasiewicz's implication, which, if added, would allow to get the full-fledged Ł4, it does not agree with our informal intuition about the values of *RS*. For instance, consider the case when $\mathbf{2/3}$ was assigned to the antecedent, while the consequent has the value $\mathbf{1/3}$. In terms of *RS* logic values this assignment means **T0** for $\mathbf{2/3}$ and **F0** for $\mathbf{1/3}$. Under these circumstances the value of Łukasiewicz's implication will be $\mathbf{2/3}$, that looks at least strange.

Third, this logic is in a sense an epistemic one. It can be applied as a logical tool for public announcement modelling and other exciting ventures.

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ADDITION

Equality of consequence relations in finite-valued logical matrices

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abstract. In this paper the procedure is presented that allows to determine in finite number of steps if consequence relations in two finite-valued logical matrices for propostional language L are equal.

Keywords: product of logical matrices, consequence relation, equality of matrices

In his paper 'A test for the equality of truth-tables' [2], J. Kalicki has described a general method for testing the equality of the classes of tautologies in different finite-valued matrices. Below I present a generalization of Kalicki's method which allows to test whether the consequence relations in two finite-valued logical matrices are equal.

First, the question of equality of consequence relations in two arbitrary matrices will be reduced to the question of the properties of a single matrix. This matrix will be obtained from initial matrices via the operation of product, but it will have four classes of truth-values instead of the standard two (designated and non-designated). On the basis of these four classes I will define several consequence relations. The properties that these relations display in the product matrix will define if two initial matrices are equal in terms of consequence relation. Then I will show that it is sufficient to consider a finite set of formulas to investigate the properties in question, and that therefore a finite number of steps is required to determine if consequence relations are equal in two finite-valued matrices.

Let us begin with some necessary definitions.

Definition 1. A logical matrix is a structure $\mathfrak{M} =< V, F, D >$, where V is the set of truth-values, F is a set of functions on V called *basic functions*, and D is a *designated* subset of V.

In this paper we will only consider the logical matrices where V is finite.

If for any n it is true that $\mathfrak{M}$ contains as much n-ary elements of F as there are n-ary connectives in some propositional language L, $\mathfrak{M}$ is a logical matrix for L. In that case we can establish a one-to-one correspondence between the elements of F and the connectives of L, and define a valuation of a formula in $\mathfrak{M}$.

Definition 2. A valuation v of formula A in $\mathfrak{M}$ is a homomorphism of L in $< V, F >$ such that

1. if A is a propositional variable, then $v(A) \in V$;
2. if $A_1, A_2, \cdots, A_n$ are formulas, and $\mathbb{C}$ is an n-ary connective of L, then $v(\mathbb{C}(A_1, A_2, \cdots, A_n)) = f^n(v(A_1), v(A_2), \cdots, v(A_n))$, where f^n is a function from F corresponding to $\mathbb{C}$.

The definition of consequence relation in $\mathfrak{M}$ is a standard one.

Definition 3. $\Gamma \vDash (\mathfrak{M})B$ iff there is no valuation v in $\mathfrak{M}$, such that $v[\Gamma] \subseteq D(\mathfrak{M})$ (i.e. every formula from Γ assumes a truth-value designated in $\mathfrak{M}$), and $v(A) \notin D(\mathfrak{M})$.

Let us denote as $C(\mathfrak{M})$ a set of ordered pairs $< \Gamma, B >$, such that Γ is a set of formulas, B is a formula, and $\Gamma \vDash (\mathfrak{M})B$. Now we will define the equality of consequence relations in two arbitary matrices for L.

Definition 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the matrices for L. The consequence relations in $\mathfrak{A}$ and $\mathfrak{B}$ are equal iff $C(\mathfrak{A}) = C(\mathfrak{B})$.

Now we will make the transition from two matrices to one by applying the product operation. If $\mathfrak{A}$ and $\mathfrak{B}$ are the matrices for L, a one-to-one correspondence between the elemnts of their sets of basic functions can be established. This allows us to give the following definition.

Definition 5. A product of matrices $\mathfrak{A}$ and $\mathfrak{B}$ is a matrix $\mathfrak{C} = \mathfrak{A} \otimes \mathfrak{B}$, such that

- $V(\mathfrak{C})$ is a Cartesian product of $V(\mathfrak{A})$ and $V(\mathfrak{B})$;
- for each pair pair of mutually corresponding k-ary basic functions $f^k(x_1, x_2, \cdots, x_k)$ from $\mathfrak{A}$ and $g^k(y_1, y_2, \cdots, y_k)$ from $\mathfrak{B}$ there is one and only one basic operation h^k from $\mathfrak{C}$, and $h^k(< x_1, y_1 >, < x_2, y_2 >, \cdots, < x_k, y_k >) = < f^k(x_1, x_2, \cdots, x_k), g^k(y_1, y_2, \cdots, y_k) >$.

This is a standard product operation. However, the truth-values in $\mathfrak{C}$ will be divided into four classes[1]:

- $< x_i, y_j > \in \omega(\mathfrak{C})$ iff $x_i \in D(\mathfrak{A})$ and $y_j \in D(\mathfrak{B})$;
- $< x_i, y_j > \in \xi(\mathfrak{C})$ iff $x_i \in D(\mathfrak{A})$ and $y_j \notin D(\mathfrak{B})$;
- $< x_i, y_j > \in \xi'(\mathfrak{C})$ iff $x_i \notin D(\mathfrak{A})$ and $y_j \in D(\mathfrak{B})$;
- $< x_i, y_j > \in \phi(\mathfrak{C})$ iff $x_i \notin D(\mathfrak{A})$ and $y_j \notin D(\mathfrak{B})$.

I will now consider two definitions of consequense relation based on these four classes, $\vDash_\cup$ and $\vDash_\cap$.

Definition 6. $\Gamma \vDash_\cup (\mathfrak{C})B$ iff there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \in \phi(\mathfrak{C})$.

Lemma 1. $\Gamma \vDash_\cup (\mathfrak{C})B$ iff $\Gamma \vDash (\mathfrak{A})B$ or $\Gamma \vDash (\mathfrak{B})B$.

Proof. *(i)* Let $\Gamma \vDash_\cup (\mathfrak{C})B$, and $\Gamma \nvDash (\mathfrak{A})B$, and $\Gamma \nvDash (\mathfrak{B})B$. Then there exists a valuation v^* in $\mathfrak{A}$, such that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$, and there exists a valuation u^* in $\mathfrak{B}$, such that $u^*[\Gamma] \subseteq D(\mathfrak{B})$ and $u^*(A) \notin D(\mathfrak{B})$. For every v and u there is a mapping w of the propositional variables of L on $V(\mathfrak{A}) \times V(\mathfrak{B})$, such that $w(p_k) =< v(p_k), u(p_k) >$, where p_k is a propositional variable. Obviously, every such w is a valuation in $\mathfrak{C}$. By definition of $\mathfrak{C}$, w^* obtained from v^* and u^* is such a valuation that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \phi(\mathfrak{C})$. That contradicts our assumption.

(ii) Let $\Gamma \nvDash_\cup (\mathfrak{C})B$, and $\Gamma \vDash (\mathfrak{A})B$ or $\Gamma \vDash (\mathfrak{B})B$. Then there is a valuation w^* in $\mathfrak{C}$, such that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \phi(\mathfrak{C})$. For

[1] This is essentially a distribution introduced by Kalicki [2], but he only needed three classes, so elements of $\xi(\mathfrak{C})$ and $\xi'(\mathfrak{C})$ were assigned to the same class.

every valuation w in $\mathfrak{C}$ there is the following valuation v in $\mathfrak{A}$: if $w(p_k) =< x_i, y_j >$, then $v(p_k) = x_i$. By definition of $\mathfrak{C}$, v^* obtained this way from w^* is such a valuation in $\mathfrak{A}$ that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$. The reasoning for valuation u^* in $\mathfrak{B}$ is analogous, and leads to the contradiction. $\square$

Definition 7. $\Gamma \vDash_\cap (\mathfrak{C})B$ iff all three of the following conditions are fulfilled:

- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$;
- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$;
- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

Lemma 2. $\Gamma \vDash_\cap (\mathfrak{C})B$ iff $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$.

Proof. *(i)* Let $\Gamma \vDash_\cap (\mathfrak{C})B$, and $\Gamma \nvDash (\mathfrak{A})B$, and $\Gamma \nvDash (\mathfrak{B})B$. The reasoning is analogous to the one in Lemma 1.

(ii) Let $\Gamma \vDash_\cap (\mathfrak{C})B$, and either $\Gamma \nvDash (\mathfrak{A})B$ or $\Gamma \nvDash (\mathfrak{B})B$. Suppose $\Gamma \nvDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$. Then there is a valuation v^* in $\mathfrak{A}$, such that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$. Now we have to consider two possibilities.

(ii.1) There is a valuation u^* in $\mathfrak{B}$, such that $u^*[\Gamma] \subseteq D(\mathfrak{B})$ and $u^*(A) \in D(\mathfrak{B})$. In this case, from v^* and u^* we can obtain a corresponding valuation w^* in $\mathfrak{C}$ (see Lemma 1), such that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \xi'(\mathfrak{C})$. But then $\Gamma \nvDash_\cap (\mathfrak{C})B$, which contradicts our assumption.

(ii.2) For every valuation u in $\mathfrak{B}$, $u[\Gamma] \notin D(\mathfrak{B})$. Let u' be such a valuation that $u'[\Gamma] \notin D(\mathfrak{B})$, and $u'(A) \notin D(\mathfrak{B})$. The corresponding valuation w' in $\mathfrak{C}$ obtained from v^* and u' in the same way as in Lemma 1 will be such that $w'[\Gamma] \subseteq \xi(\mathfrak{C})$, and $w'(A) \in \phi(\mathfrak{C})$. Let u'' be such a valuation that $u''[\Gamma] \notin D(\mathfrak{B})$, and $u''(A) \in D(\mathfrak{B})$. The corresponding valuation w'' in $\mathfrak{C}$ obtained from v^* and u'' will be such that $w''[\Gamma] \subseteq \xi(\mathfrak{C})$, and $w''(A) \in \xi'(\mathfrak{C})$. Both cases lead us to the contradiction with the assumption that $\Gamma \vDash_\cap (\mathfrak{C})B$.

The reasoning for $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \nvDash (\mathfrak{B})B$ is analogous.

(iii) Let $\Gamma \nvDash_{\cap} (\mathfrak{C})B$, and $\Gamma \vDash (\mathfrak{A})B$, and $\Gamma \vDash (\mathfrak{B})B$. If $\Gamma \nvDash_{\cap} (\mathfrak{C})B$, three cases are possible:

(iii.1) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$;

(iii.2) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$;

(iii.3) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

The reasining for all three cases is the same. We obtain from w the corresponing valuations v in $\mathfrak{A}$ and u in $\mathfrak{B}$ in the same way as we did in Lemma 1. Due to the properties of w described in *(iii.1)–(iii.3)*, either v, or u, or both of them will be such that they will lead to the contradiction with the assumption that $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$. □

From Lemma 1 we have that $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{A}) \cup C(\mathfrak{B})$. From Lemma 2 we have that $C(\mathfrak{C}, \vDash_{\cap}) = C(\mathfrak{A}) \cap C(\mathfrak{B})$. Also, we have that $C(\mathfrak{A}) = C(\mathfrak{B})$ iff $C(\mathfrak{A}) \cup C(\mathfrak{B}) = C(\mathfrak{A}) \cap C(\mathfrak{B})$. Therefore, $C(\mathfrak{A}) = C(\mathfrak{B})$ iff $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{C}, \vDash_{\cap})$.

Now let us consider another consequence relation.

Definition 8. $\Gamma \vDash^{*} (\mathfrak{C})B$ iff either

- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$,
- and there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$,
- and there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$,
- or there is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \in \phi(\mathfrak{C})$.

Lemma 3. $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{C}, \vDash_{\cap})$ iff $\Gamma \vDash^{*} (\mathfrak{C})B$ for each set of formulas Γ and each formula B.

Proof. If $C(\mathfrak{C}, \models_\cup) = C(\mathfrak{C}, \models_\cap)$, for each Γ and B it is true that either $\Gamma \models_\cap (\mathfrak{C})B$ or $\Gamma \not\models_\cup (\mathfrak{C})B$. Both cases lead to $\Gamma \models^* (\mathfrak{C})B$. Now let us assume that $\Gamma \models^* (\mathfrak{C})B$ for some arbitrary Γ and B. Then *(i)* for every evaluation w in $\mathfrak{C}$, if $w[\Gamma] \subseteq \omega(\mathfrak{C})$ then $w(A) \in \omega(\mathfrak{C})$, if $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, then $w(A) \in \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, if $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cap \xi'(\mathfrak{C})$, then $w(A) \in \omega(\mathfrak{C}) \cap \xi'(\mathfrak{C})$, or *(ii)* there is at least one valuation in $\mathfrak{C}$, such that all formulas from Γ assume a truth value from $\omega(\mathfrak{C})$, and B assumes a value from $\phi(\mathfrak{C})$. In the first case $\Gamma \models_\cap (\mathfrak{C})B$. In the second case $\Gamma \not\models_\cup (\mathfrak{C})B$. Therefore $C(\mathfrak{C}, \models_\cup) = C(\mathfrak{C}, \models_\cap)$. □

Below, the number of formulas that need to be considered will be narrowed down to a finite set. I will use the method proposed by J. Kalicki in [1] with necessary modifications.

Lemma 4. For each matrix $\mathfrak{C}_m$, where m is the number of the elements of $V(\mathfrak{C})$, the following is true: if for each pair Γ and B that contains $i \leq m$ different variables $\Gamma \models^* (\mathfrak{C}_m)B$, then for each pair Δ and E that contains $m + t(t = 0, 1, \cdots)$ different variables $\Delta \models^* (\mathfrak{C}_m)E$.

Proof. Let us use the induction by t. For $t = 0$ it is obvious that for each Γ and B that contains $i \leq m$ different variables $\Gamma \models^* (\mathfrak{C}_m)B$, then for each pair$\Delta$ and E that contains m different variables $\Delta \models^* (\mathfrak{C}_m)E$.

Let us assume that the theorem is true for $t \leq k$ and prove it for $t = k + 1$. Let there exist Δ and E that contain $m + k + 1$ different variables, and $\Delta \not\models^* (\mathfrak{C}_m)E$. Then there exists a valuation w_0 in $\mathfrak{C}_m$ that maps the variables $p_1, p_2, \cdots, p_{m+k+1}$ on values $x_1, x_2, \cdots, x_{m+k+1}$ respectively, such that either *(i)* $w_0[\Delta] \subseteq \omega(\mathfrak{C})$, and $w_0(E) \notin \omega(\mathfrak{C})$, or *(ii)* $w_0[\Delta] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(E) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, or *(iii)* $w_0[\Delta] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(E) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

Let us consider (i). Due to the fact that in $\mathfrak{C}_m$ there is m different truth-values in total, there will be at least two $i_1 \neq i_2$ among $i = 1, 2, \cdots, m + k + 1$, such that $x_{i_1} = x_{i_2}$. Now let us consider Δ' and E', obtained from Δ and D by replacement of all instances of p_{i_2} with p_{i_1}. It is clear that $w_0[\Delta'] \subseteq \omega(\mathfrak{C}_m)$ and $w_0(E') \notin \omega(\mathfrak{C}_m)$. Because Δ' and E' contain $m + k$ different variables, according to the inductive assumption, $\Delta' \models^* (\mathfrak{C}_m)E'$. Therefore, there exists a valuation w^* in

$\mathfrak{C}_m$, which maps the variables $p_1, p_2, \cdots, p_{i_2-1}, p_{i_2+1}, \cdots, p_{m+k+1}$ on the values $y_1, y_2, \cdots, y_{i_2-1}, y_{i_2+1}, \cdots, y_{m+k+1}$ respectively, such that $w^*[\Delta'] \subseteq \omega(\mathfrak{C}_m)$ and $w^*(E') \in \phi(\mathfrak{C}_m)$. In this case we can construct a valuation w^{**}, which maps the variables $p_1, p_2, \cdots, p_{m+k+1}$ on the values $y_1, y_2, \cdots, y_{i_2-1}, y_{i_1}, y_{i_2+1}, \cdots, y_{m+k+1}$ respectively. It is clear that $w^{**}[\Delta] \subseteq \omega(\mathfrak{C}_m)$ and $w^{**}(E) \in \phi(\mathfrak{C}_m)$. But then $\Delta \models^* (\mathfrak{C}_m)E$, which contradicts our assumption.

The reasoning for (ii) and (iii) is analogous. □

For m different variables there is $k = m^m$ different valuations $v_1, v_2, \cdots, v_k$ in $\mathfrak{C}_m$. We can assign to each variable $p_i (1 \leq i \leq m)$ a unique value-sequence $|p_i| =< x_1, x_2, \cdots, x_k >$, where $x_l = v_l(p_i)(1 \leq l \leq k)$.

Now let us construct the following sequence of the classes of formulas:

- The elements of CL_0 are the variables $p_1, p_2, \cdots, p_m$ exclusively;
- to a class CL_{t+1} belong all formulas that can be constructed by means of one connective, an element of class CL_t, and (if needed) elements of $CL_{n \leq t}$.

For each formula B from CL_n we can calculate the corresponding value-sequence $|B| =< y_1, y_2, \cdots, y_k >$, where $y_j (1 \leq j \leq k)$ is obtained from j-th elements of sequences assigned to the variables included in B. Let us denote the set of value-sequences for elements of Cl_n as $|Cl_n|$. Because the sequences in question consist of k elements, and the number of truth-values equals m, in total there is m^k possible sequences. Therefore, there is a finite $n_0 \leq m^k$, such that $|CL_{n_0}|$ contains no value-sequence which is not also the element of some $|CL_{n<n_0}|$.

LEMMA 5. The value-sequence of any formula $B \in CL_{n>n_0}$ is identical to some element of $|CL_{n<n_0}|$.

PROOF. Let $B \in CL_{n_0+1}$. By definition of CL_{n_0+1}, formula B consists of the main connective, at least one formula from CL_{n_0}, and probably elements of $CL_{n_i<n_0}$. By definition of n_0, each value-sequence from $|CL_{n_0}|$ is also present in some $|CL_{n_j<n_0}|$. Therefore, by definition of $|CL|$, there is a set $|CL_{max(i,j)+1}|$, which contains the value-sequence identical to $|B|$. Because $n_i < n_0$ and $n_j < n_0$, we have that $max(n_i, n_j) + 1 \leq n_0$, $|B| \in |CL_{n \leq n_0}|$. From that, according to the definition of n_0, we obtain that $|B| \in |CL_{n<n_0}|$. The theorem is proved for CL_{n_0+1}. The generalization for $CL_{n>n_0}$ is obvious. □

So the set $|CL_1| \cup |CL_2| \cup \cdots \cup |CL_{n_0}|$ contains all value-sequences possible in $\mathfrak{C}_m$ for formulas that contain no more than m different variables. From this fact and Lemma 4 it follows that $\Gamma \models^* (\mathfrak{C}_m)B$ for each Γ and B iff $\Delta \models^* (\mathfrak{C}_m)E$ for every Δ and E that consist exclusively of the elemnts of $CL_1 \cup CL_2 \cup \cdots \cup CL_{n_0}$.

This concludes the construction of the procedure for testing if $C(\mathfrak{A}) = C(\mathfrak{B})$ for two arbitrary finite-valued matrices $\mathfrak{A}$ and $\mathfrak{B}$ for some propositional language L.

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Generalization of Kalmar's method for quasi-matrix logic[1]

Yuriy V. Ivlev

abstract. Quasi-matrix logic is based on the generalization of the principles of classical logic: *bivalency* (a proposition take values from the domain $\{t$ (truth), f (falsity)$\}$); *consistency* (a proposition can not take on both values); *excluded middle* (a proposition necessarily takes some of these values); *identity* (in a complex proposition, a system of propositions, an argument the same proposition takes the same value from domain $\{t, f\}$); *matrix principle* — logical connectives are defined by matrices. As a result of our generalization, we obtain quasi-matrix logic principles: the principle of *four-valency* (a proposition takes values from domain $\{t^n, t^c, f^c, f^i\}$) or *three-valency* (a proposition takes values from domain $\{n, c, i\}$); *consistency*: a proposition can not take more than one value from $\{t^n, t^c, f^c, f^i\}$ or from $\{n, c, i\}$; the principle of *excluded fifth or fourth*; *identity* (in a complex proposition, a system of propositions, an argument the same proposition takes the same value from domain $\{t^n, t^c, f^c, f^i\}$ or domain $\{n, c, i\}$); the *quasi-matrix principle* (logical terms are interpreted as quasi-functions). Quasi-matrix logic is a logic of factual modalities.

Keywords: quasi-matrix logic, semantic completeness, decision problem, Kalmar's method

1 Kalmar's method

Well-known proof method for methateorem of semantic completeness of classical propositional calculus, which may be also treated as an approach to the solution of the decision problem, implies the proof of the following lemma:

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Lemma 1. Assuming that D is a formula, $a_1, \ldots, a_n$ are all different variables, occurring in D, $b_1, \ldots, b_n$ are truth-values of these variables; let A_i be $a_i, \neg a_i$, depending on whether b_i takes value t or f; let D' be D or $\neg D$ depending on whether D takes value t or f with truth-values $b_1, \ldots, b_n$ variables $a_1, \ldots, a_n$. Then $A_1, \ldots, A_n \Rightarrow D'$.

($\Rightarrow$ is here a sign (symbol) for logical entailment, $\neg$ — for negation, t и f — truth and falsity, respectively.)

2 Generalization of Kalmar's method for many-valued matrix logic

At the end of the sixties of the 20-th century I was able to generalize this method for functionally complete many-valued matrix logics. (Probably the generalization of this kind had been done earlier by somebody else, but I have not heard of it up to now.)

Let's illustrate the basic principles underlying the generalization with one of the system of modal logic Sb^- constructed by me.

Logical terms of language: $\neg$, $\supset$, $\Box$, $\Diamond$. ('$\supset$', '$\Box$', '$\Diamond$' — are respectively signs for implication, necessity and possibility)

2.1 Semantics Definitions of logical terms

$\supset$	t^n	t^c	f^i	f^c
t^n	t^n	t^c	f^i	f^c
t^c	t^n	t^c	f^c	f^c
f^i	t^n	t^n	t^n	t^n
f^c	t^n	t^c	t^c	t^c

A	$\neg A$	$\Box A$	$\Diamond A$
t^n	f^i	t^n	t^n
t^c	f^c	f^c	t^c
f^i	t^n	f^i	f^i
f^c	t^c	f^c	t^c

t^n, t^c, f^c, f^i — are respectively truth-values 'necessary truth', 'contingent truth', 'contingent falsity', 'necessary falsity'. Designated values are t^n and t^c.

2.2 Formalisation

The calculus includes schemes of axioms of classical propositional calculus, **modus ponens** rule of inference and also following schemes of axioms:

$\Box A \supset A$; $\neg\Box\neg A \supset \Diamond A$; $\Diamond A \supset \neg\Box\neg A$; $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$; $\Diamond B \supset \Diamond(A \supset B)$; $\Diamond\neg A \supset \Diamond(A \supset B)$; $\Diamond(A \supset B) \supset$

$(\Box A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

For the proof of meta-theorem of semantic completeness of calculi Sb^- the following **lemma** is needed.

LEMMA 2. Assuming that D is a formula, $a_1, ..., a_n$ are all different variables occurring in D, $b_1, ..., b_n$ are truth-values of these variables. Let A_i be $\Box a_i$, $a_i \& \Diamond\neg a_i$, $\neg a_i \& \Diamond a_i$, $\neg\Diamond a_i$ depending on whether b_i is t^n, t^c, f^c or f^i. Let D' be $\Box D$, $D \& \Diamond\neg D$, $\neg D \& \Diamond D$ or $\neg\Diamond D$ depending on whether D takes value t^n, t^c, f^c or f^i with truth-values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$. Then $A_1, ..., A_n \Rightarrow D'$. (& — is here a sign for conjunction)

Lemma is proved by the use of recurrent mathematical induction.

If formula D takes designated value with all possible truth-values of its variables, then D' is $\Box D$ or $D \& \Diamond\neg D$. In each case $A_1, ..., A_n \Rightarrow D$.

Let us substitute assumption $\Box a_{i+1}$ with number $i+1$ from the set of assumptions $A_1, ..., A_n$ for the set of formulas $a_{i+1}, \neg\Diamond\neg a_{i+1}$, assumption $a_{i+1} \& \Diamond\neg a_{i+1}$ for the set of formulas $\Diamond\neg a_{i+1}, a_{i+1}$, assumption $\neg a_{i+1} \& \Diamond a_{i+1}$ for the set of formulas $\neg a_{i+1}, \Diamond a_{i+1}$, assumption $\neg\Diamond a_{i+1}$ for the set of formulas $\neg a_{i+1}, \neg\Diamond a_{i+1}$. Then all assumptions with number $i+1$ may be eliminated.

2.3 Illustration

1. $A_1, ..., A_i, a_{i+1}, \neg\Diamond\neg a_{i+1} \Rightarrow D$,
2. $A_1, ..., A_i, a_{i+1}, \Diamond\neg a_{i+1} \Rightarrow D$,
3. $A_1, ..., A_i, \neg a_{i+1}, \Diamond a_{i+1} \Rightarrow D$,
4. $A_1, ..., A_i, \neg a_{i+1}, \neg\Diamond a_{i+1} \Rightarrow D$,
5. $A_1, ..., A_i, a_{i+1} \Rightarrow D$ – from 1, 2,
6. $A_1, ..., A_i, \neg a_{i+1} \Rightarrow D$ – from 3, 4,
7. $A_1, ..., A_i \Rightarrow D$ – from 5, 6.

In my doctoral thesis I brought forward 30 problems calling for solution. Later these ideas were published in monograph [8, p. 208–

217]. Many of these problems have been solved by now. The solutions were published in 13 PhD theses and publications. Some of the problems have not been solved yet. One of these problems (problem number 9) may be formulated as follows: *if logic is functionally complete, then for any propositional variable a and any truth value i there is a formula $f_i(a)$ containing only this variable and taking some designated value if and only if a takes value i; suppose $a_1, a_2, ..., a_n$ are all different variables occurring in D; suppose $b_1, b_2, ..., b_n$ are the truth-values of these variables; suppose A_s is $f_k(a_s)$, if b_s is k; suppose D' is $f_r(D)$ ($f_r(D)$ is a formula formed on the basis of D and taking designated value with truth-values $b_1, b_2, ..., b_n$ of the variables $a_1, a_2, ..., a_n$). Then $A_1, A_2, ..., A_n \Rightarrow D$.*

For example, 1, $\frac{1}{2}$, 0 are the truth-values of three-valued modal logic of Łukasiewicz; $f_1(a)$ is $\Box a$, $f_{\frac{1}{2}}(a)$ is $\Diamond a \& \Diamond \neg a$, $f_0(a)$ is $\neg \Diamond a$; if formula takes value 1 with some truth-values of its variables, then $f_r(D)$ is $\Box D$, etc.; assumptions may be eliminated like it was stated for Sb^-.

The ninth problem is the problem of finding the proof for metatheorem of semantic completeness of all known finite-valued matrix logics and finding sets of axioms for all logics of this kind stated semantically.

The seventeenth problem is the problem of generalization of this method for the proof of semantic completeness (and solution of the decision problem) of propositional quasi-matrix logics. This problem has not been solved for a long time. The solution is brought off in this article.

3 Quasi-matrix logic

Quasi-matrix is a set $(Q, G, qf_1, ..., qf_s)$, where Q and G are nonempty sets such that $Q \subseteq G$; $qf_1, ..., qf_s$ are quasi-functions.

If a function is a correspondence in virtue of which an object from some (functional) domain is related with certain object (from the range of the function) then a quasi-function is a correspondence in virtue of which an object from a certain subset of some set is related with some object from a certain subset of some or another set (from the range of the quasi-function).

3.1 Examples

Function: $\{(a,d),(b,k),(c,k)\}$.

Quasi-function: $\{(a,d)\veebar_2(a,k),(c,m)\} = \{\{(a,d),(c,m)\}\veebar_2\{(a,k),(c,m)\}\}$,

Quasi-function: $\{\veebar_4((a,k),(a,n),(c,k),(c,n)),(d,r) = \veebar_4[\{(a,k),(d,r)\},\{(a,n),(d,r)\},\{(c,k),(d,r)\},\{(c,n),(d,r)\}]\}$,

$\veebar_2$ and $\veebar_4$ are two- and four-place (respectively) metalinguistic exclusive disjunctions. Let us assume that disjunction may be degenerative, i. e. in this particular case quasi-function is just a function. Then a matrix is a particular case of quasi-matrix.

In the general case an object of application of a quasi-function, as well as truth-value of a quasi-function, are indefinite. Only sub-range of the range of quasi-function, which includes this object, and sub-range of the range of values of a quasi-function, which contains a value of a quasi-function, are defined.

Such vagueness may be of a cognitive nature. It takes place, when the above-mentioned correspondence or relation is objectively functional, but this is not known to the researcher. For example, there are three probable variants of translation of a certain word in a dictionary, but the translator doesn't know, which of these three readings is the most appropriate in the present case (context). Such situations also appear in systems of automatic translation.

Another cause of indetermination is that reality may be indeterminate itself. For example, for planning of a production we have to take into account the following reasons. Suppose that we know the limits of alteration of a quantity of raw stuff, which will be factored next year. But it s impossible to figure out any rigid link between definite quantity of a factored raw stuff and a quantity of output, even if we knew a quantity of man-power, equipment etc.

For the first time some particular examples of quasi-functions were represented by H. Reichenbach (1932, 1935, 1936), Z. Zavarski (1936), F. Gonseth (1938, 1941), N. Rescher (1962, 1964, 1965, 1969). Rescher considers a material implication and defines it as follows:

A	$\supset$	B
t	t	t
t	f	f
f	(t, f)	t
f	(t, f)	f

(t, f) is not a determinate truth-values. This bracketed entry (t, f) means that either one of these two truth-values may occur in the various particular cases. Hence, depending on specific sense of propositions, the whole implication may be either true or false. Other logical terms are formulated in a usual way.

It is obvious that not all tautologies of a classical propositional logic of the form $A \supset B$ take the truth-value 't' under any given assignment of truth-values to elementary propositions.

Rescher formulates the conception of quasi-tautology. He adopts t and (t, f) in his quasi-functional system Q as designated truth-values. Then quasi-tautology is a formula which invariably does or can take either of this designated truth-values for every assignment of truth-values to its propositional variables. But if we bring to a logical end Rescher's reasoning we also have to treat as a quasi-tautology propositional variable p.

Then Rescher 'corrects' definitions of Łukasiewicz' three-valued logic.

A	$\&$	B
$\frac{1}{2}$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$

Independently of the above-mentioned and some other authors I came to the same considerations at the end of the sixties / beginning of the seventies. My ideas were concerned with the way of modal logic development. Though by that time a lot of different 'logical systems' had been constructed, it wasn't clear, what kind of modal operators and notions (either factual or logical necessity, possibility etc.) were defined by these systems. It made the application of modal systems to the natural reasoning analysis very difficult. This condition of modal logic seemed to me unsatisfactory and inadequate. On purpose to overcome these difficulties I distinguished two different branches of modal logical investigations: proper logic (or logic itself) and an imitation of logic. Proper logic deals with the forms of thoughts. H. Curry called this kind of logic a philosophical one. Imitation of logic is a certain (formal) system, e. g. algebraic system, which in some respect resembles philosophical logic (usually with respect to some technical symbols and signs) [15].

In the following explanations I am treating modern logic as a philosophical logic in the sense of Curry.

In logic, as well as in each other science, it's possible to distinguish empirical and theoretical levels of development. An essential feature of a theory is its ability to explain phenomena. As I think, my approach to the analysis of logical modalities, elaborated by N. Arkhiereev, possesses this ability. Theory of factual modalities, which is to be based on quasi-matrix logic, has not been yet completely developed. (Fundamental ideas of theory of logical modalities are represented in [1, 2, 6, 7, 13, 14].)

I began to work out quasi-matrix logic with constructing the system of minimal modal logic.

3.2 Minimal modal logic S_{min}

(Symbols of formalised language: $\Box, \Diamond, \neg, \supset$).

Łukasiewicz's well-known statement about impossibility of proper definitions of modal operators 'necessary ($\Box$) and 'possibly' ($\Diamond$) in terms of 'truth' and 'falsity' is valid only if these operators are interpreted as functions.

But if we interpret modal operators as quasi-functions, it becomes possible to define them in above-mentioned terms.

Let's consider formula $\Box A$. Assume A takes value f (falsehood). Then formula $\Box A$ also takes value f, since not-existing state of affairs can not be necessary (both logically and factually). Assume formula A takes value t (truth). What truth-value takes formula $\Box A$ in this case? The value is indeterminate. Formula $\Box A$ takes either value t, or value f. Let's notify this situation by t/f.

By the same reasoning, we can conclude that truth-value of the formula $\Diamond A$ is indeterminate, when formula A takes value f. Definitions of signs of negation and implication are usual. Designated truth-value is t.

Principles of classical propositional logic and logic S_{min}

Classical propositional logic principles	**Principles of quasi-matrix logic S_{min}**
(1) the principle of bivalency (propositions take values from the domain $\{t$ (truth), f (falsity)$\}$)	the principle of bivalency
(2) the principle of consistency (a proposition can not have both the values)	the principle of consistency
(3) the principle of excluded middle (a proposition necessarily has some of these values)	the principle of excluded middle
(4) the principle of identity (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{t, f\}$)	the principle of identity
(5) the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in classical logic this principle acts as a matrix principle — logical connectives are interpreted as functions)	the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in S_{min} this principle acts as a quasi-matrix principle — logical terms are interpreted as quasi-functions)

S_{min} — formalism which is adequate to the system constructed semantically. S_{min}-calculus is an extension of a classical propositional calculus with added new axiom schemes: $\Box A \supset A$, $A \supset \Diamond A$.

S_{min}-calculus is weaker than basic modal logic of Łukasiewicz, since the formula $\Box A \equiv \neg\Diamond\neg A$ is not provable there.

For the proof of semantic completeness meta-theorem of S_{min}-calculus, we define alternative interpretation as follows.

Alternative interpretation is a function $||\ ||$ such as to: If P is — propositional variable then $||P|| \in \{t, f\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = t \Leftrightarrow ||A|| = f$; $||A \supset B|| = f \Leftrightarrow ||A|| = f$ or $||B|| = t$; $||A|| = f \Rightarrow ||\Box A|| = f$; $||A|| = t \Rightarrow ||\Box A|| \in \{t, f\}$; $||A|| = t \Rightarrow ||\Diamond A|| = t$; $||A|| = f \Rightarrow ||\Diamond A|| \in \{t, f\}$. ($\Leftrightarrow$ and $\Rightarrow$ are here abbreviations for expression 'if and only if' ('iff') and 'if..., then...' respectively.)

Formula is satisfiable iff it takes the value 'true' in some alternative interpretation. Formula is valid iff it is true under each alternative interpretation.

3.3 Four-valued quasi-matrix logical systems

Truth-values t^n, t^c, f^c, f^i are interpreted as follows: proposition taking values t^n describes a state of affairs which takes place in reality and which is strictly determined by certain circumstances; proposition taking values t^c describes a state of affairs which takes place in reality and which is not strictly determined by either circumstances; proposition taking values f^c describes a state of affairs which doesn't exist in reality and the absence of which is not strictly determined by either circumstances; proposition taking values f^i describes a state of affairs which doesn't exist in reality and which absence is strictly determined by certain circumstances.

Four-valued quasi-matrix logic based on the following generalization of classical logic principles.

Classical logic principles	**Quasi-matrix logic principles**
(1) the principle of <u>bivalency</u> (propositions take values from the domain $\{t$ (truth), f (falsity)$\}$)	the principle of <u>four-valency</u> (propositions take values from the domain $\{t^n, t^c, f^c, f^i\}$)
(2) the principle of <u>consistency</u> (a proposition can not have both the values)	<u>consistency</u>: can not have more than one value from $\{t^n, t^c, f^c, f^i\}$
(3) the principle of <u>excluded middle</u> (a proposition necessarily has some of these values)	the principle of <u>excluded fifth</u>
(4) the principle of <u>identity</u> (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{t, f\}$)	<u>identity</u> from the domain $\{t^n, t^c, f^c, f^i\}$

(5) the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in propositional logic this principle acts as a matrix principle — logical connectives are defined by matrices, in predicate logic it shows up in the interpretation of logical terms and predicates as truth functions).	the quasi-matrix principle (logical terms are interpreted as quasi-functions)

Logical terms are the same as those in the S_{min}-system.

Definitions of logical terms:

A	$\neg A$	**a**		**b**		**c**		**d**		**e**	
		$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$
t^n	f^i	t	t	t^n	t^n	t^n	t^n	t^c	t^c	t^c	t^c
t^c	f^c	f	t	f^c	t^c	f^i	t^n	f^c	t^c	f^i	t^n
f^i	t^n	f	f	f^i	f^i	f^i	f^i	f^c	f^c	f^c	f^c
f^c	t^c	f	t	f^c	t^c	f^i	t^n	f^c	t^c	f^i	t^n

A	$\neg A$	**f**		**g**		**h**		**i**	
		$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$
t^n	f^i	t	t	t	t	t^n	t^n	t^c	t^c
t^c	f^c	f^i	t^n	f^c	t^c	f	t	f	t
f^i	t^n	f	f	f	f	f^i	f^i	f^c	f^c
f^c	t^c	f^i	t^n	f^c	t^c	f	t	f	t

(−)

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	t^c	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	t^c	t^c	t^c

( )

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	$t^n \vert t^c$	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	t^c	t^c	$t^n \vert t^c$

(+)

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	$t^n \vert t^c$	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	$t^n \vert t^c$	t^c	$t^n \vert t^c$

t and $t^n|t^c$ mean «either t^n, or t^c». f and $f^i|f^c$ mean «either f^i, or f^c».

Following logical systems have been constructed on the basis of above-stated definitions: Sa^-, Sa, Sa^+, Sb^-, Sb, Sb^+, Sc^-, Sc, Sc^+, Sd^-, Sd, Sd^+, Se^-, Se, Se^+, Sf^-, Sf, Sf^+, Sg^-, Sg, Sg^+, Sh^-, Sh, Sh^+, Si^-, Si, Si^+. Lower case letters occurring in the name of systems corresponds to the definition of modal terms, signs $+, -$ and their absence correspond to the definition of implication.

t^n and t^c are distinguished truth-values.

The following considerations underlie the above-stated definitions of logical terms. Let us consider formula $\Box\Box A$. If the subformula A takes value t, then the value of a formula $\Box A$, as it has already been settled, is not determined, i. e. situation which is described by A takes place in reality but is determined itself either strictly or not. In the first case we have to assign to the formula $\Box A$ value t, in the second one — the value f.

I.e. in the first case a proposition A is interpreted as being true and (factually) necessary (in our terms it takes value t^n). What value in this case takes formula $\Box\Box A$? If A describes a state of affairs which is strictly determined by any circumstances, then these circumstances may in its own turn be either determined or not by some others. That is formula $\Box A$ also takes value t^n (or t^c) etc.

Such situations occur both in subjective and objective reality.

Different kinds of distinct and fuzzy determination in biology were considered by V.Yu. Ivlev in [5,6].

Semantic-constructed systems are formalized by a number of calculi including as their general part all schemes of axioms of a classical propositional calculus, modus ponens — rule of inference and following schemes of axioms: $\Box A \supset A$; $\neg\Box\neg A \supset \Diamond A$; $\Diamond A \supset \neg\Box\neg A$; $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$; $\Diamond B \supset \Diamond(A \supset B)$; $\Diamond\neg A \supset \Diamond(A \supset B)$; $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.

We sign with letter S the calculus, which is obtained from classic propositional calculus by means of above-stated eight model schemes of axioms. The calculi corresponding to the semantic-constructed systems may be worked out by addition to S of the following schemes of axioms:

Sa^-: $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$.

Sa: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$.

Sa^+: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.

Sb^-: $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Sb: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Sb^+: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Calculi $Sc^-, Sd^-, Se^-, Sf^-, Sg^-, Sh^-, Si^-$ include schemes of axioms $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$.

Calculi $Sc, Sd, Se, Sf, Sg, Sh, Si$ include schemes of axioms $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$.

Calculi $Sc^+, Sd^+, Se^+, Sf^+, Sg^+, Sh^+, Si^+$ include schemes of axioms $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$;.

Calculi, which have the same lower case letter occurring in the names (e. g. calculi Sc^-, Sc, Sc^+), differ from calculi, which have other lower case letters occurring in the names (e. g. calculi Si^-, Si, Si^+), by sets of schemes of axioms $\{\Box(A \supset B) \supset (\Diamond A \supset \Box B)\}$, $\{\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))\}$, $\{\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)\}$.

The other additional schemes of axioms of these calculi are the same:

Calculi Sc^-, Sc, Sc^+: $\Box A \supset \Box\Box A$; $\Diamond\Diamond A \supset \Diamond A$; $\Diamond\Box A \supset \Box A$; $\Diamond A \supset \Box\Diamond A$.

Calculi Sd^-, Sd, Sd^+: $\Diamond A^*$, A^* is modalized formula.

Calculi Se^-, Se, Se^+: $\Diamond\Diamond A$; $\Diamond\neg\Box A$; $\neg\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Diamond\neg\Diamond A$; $\Diamond\Box A \supset (A \supset \Box A)$; $\Diamond\Box A \supset (\Diamond A \supset A)$; $A \supset (\Diamond\neg A \supset \Box\Diamond A)$; $\neg A \supset (\Diamond A \supset \Box\Diamond A)$.

Calculi Sf^-, Sf, Sf^+: $\Diamond\Box A \supset (A \supset \Box A)$; $\Diamond\Box A \supset (\Diamond A \supset A)$; $A \supset (\Diamond\neg A \supset \Box\Diamond A)$; $\neg A \supset (\Diamond A \supset \Box\Diamond A)$.

Calculi Sg^-, Sg, Sg^+: $A \supset (\neg\Box A \supset \Diamond\Box A)$; $\neg A \supset (\Diamond A \supset \Diamond\Box A)$; $\Box\Diamond A \supset (A \supset \Box A)$; $\Box\Diamond A \supset (\Diamond A \supset A)$.

Calculi Sh^-, Sh, Sh^+: $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Box A \supset \Box\Diamond A$; $\Diamond\Diamond A \supset \Diamond A$.

Calculi Si^-, Si, Si^+: $\Diamond\Diamond A$; $\Diamond\neg\Box A$; $\neg\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Diamond\neg\Diamond A$.

We use the rule of substitution of $\neg\neg A$ with A and visa versa.

For the proof of metatheorem of semantic completeness of calculi Sb^-, Sc^-, Sd^-, Se^- (semantics for these calculi are of matrix sort) the following **lemma** is proved.

LEMMA 3. Assuming that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables. Let A_i be $\Box a_i$, $a_i \& \Diamond\neg a_i$, $\neg\Diamond a_i$, $\neg a_i \& \Diamond a_i$ depending on whether b_i is t^n, t^c, f^i or f^c. Let D' be $\Box D$, $D\&\Diamond\neg D$, $\neg\Diamond D$ or $\neg D\&\Diamond D$ depending on whether D takes value t^n, t^c, f^i or f^c with truth-values $b_1, ..., b_n$ variables $a_1, ..., a_n$. Then $A_1, ..., A_n \Rightarrow D'$.($\Rightarrow$ is here a sign for entailment.)

Lemma is proved by the use of recurrent mathematical induction.

Semantics for others calculi are quasi-matrix (proper). For the proof of metatheorem of semantic completeness of these calculi the notion of alternative interpretation is used. We have the following definition of alternative interpretation for Sa^+-system.

Alternative interpretation is a function $|| \ ||$ satisfying the following:

If P is — propositional variable then $||P|| \in \{t^n, t^c, f^i, f^c\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = t^n \Leftrightarrow ||A|| = f^i$; $||\neg A|| = t^c \Leftrightarrow ||A|| = f^c$; $||\neg A|| = f^i \Leftrightarrow ||A|| = t^n$; $||\neg A|| = f^c \Leftrightarrow ||A|| = t^c$;

$||A \supset B|| = f^c \Leftrightarrow (||A|| = t^n$ and $||B|| = f^c)$ or $(||A|| = t^c$ and $||B|| = f^i)$;

$||A \supset B|| = f^i \Leftrightarrow ||A|| = t^n$ and $||B|| = f^i$;

if either $(||A|| = t^n$ and $||B|| = t^c)$ or $(||A|| = f^c$ and $||B|| = f^i)$, then $||A \supset B|| = t^c$;

if $||A|| = f^i$ or $||B|| = t^n$, then $||A \supset B|| = t^n$;

if either $||A|| = ||B|| = t^c$ or $(||A|| = f^c$ and $||B|| = t^c)$, or $||A|| = ||B|| = f^c)$, then $||A \supset B|| \in \{t^n, t^c\}$;

$||A|| = t^n \Rightarrow ||\Box A|| \in \{t^n, t^c\}$; if either $||A|| = t^c$ or $||A|| = f^c$, or $||A|| = f^i$, then $||\Box A|| \in \{f^c, f^i\}$;

$||A|| = f^i \Rightarrow ||\Diamond A|| \in \{f^c, f^i\}$; if either $||A|| = t^n$ or $||A|| = t^c$, or $||A|| = f^c$, then $||\Diamond A|| \in \{t^n, t^c\}$.

S_r — three-valued quasi-matrix logic.

(Symbols of formalised language are the same.) n, c, i — values of S_r-system — which are interpreted respectively as 'necessary', 'contingently', 'impossibly'. State of affairs is necessary if and only if (iff) it is distinctly determined by certain circumstances; state of affairs is contingent, iff neither its existence nor its absence is not strictly determined by some circumstances; state of affairs is impossible iff its absence is strictly determined by some circumstances. Actually, here and above the evaluations of state of affairs concern (to) propositions. (To my regret, I couldn't find proper terms for evaluation of propositions.)

S_r-logic is based on the following generalizations of principles of classic logic.

Classical logic principles	**Principles of quasi-matrix logic S_r**
(1) the principle of bivalency	the principle of three-valency (propositions take values from the domain $\{n, c, i\}$)
(2) the principle of consistency	consistency: can not have more than one value from $\{n, c, i\}$
(3) the principle of excluded middle	the principle of excluded fourth
(4) the principle of identity	Identity (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{n, c, i\}$)
(5) the matrix principle	the quasi-matrix principle (logical terms are interpreted as quasi-functions)

Definitions of logical terms:

A	$\neg A$	$\Box A$	$\Diamond A$
n	i	n	n
c	c	i	n
i	n	i	i

$\supset$	n	c	i
n	n	c	i
c	n	$n \vert c$	c
i	n	n	n

$n|c$ is interpreted as 'either n or c'. n is a designated value.

Corresponding calculus includes all schemes of axioms of classical propositional calculus (note: in these schemes of axioms metasymbols A, B, C denote modalized formulas; the modalized formula definition: if A is a formula of classical propositional calculus, then $\Box A$ and $\Diamond A$ are modalized formulas; if B and C are modalized formulas, then $\Box B$, $\Diamond B$, $\neg B$, $(B\&C)$, $(B \vee C)$, $(B \supset C)$ are modalized formulas; nothing else is a modalized formula.), modus ponens, Godel's rule, all schemes of axioms of Sc^{+}-calculus, and besides the following schemes: $\Box A \supset \Diamond A$; $\neg A \supset \neg\Box A$; $\neg\Diamond A \supset \neg A$; $A \supset \Diamond A$.

Alternative interpretation is a function $|| \; ||$ for which the following helds:

If P is propositional variable then $||P|| \in \{n, c, i\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = n \Leftrightarrow ||A|| = i$; $||\neg A|| = c \Leftrightarrow ||A|| = c$; $||\neg A|| = i \Leftrightarrow ||A|| = n$;

if either $||A|| = i$ or $||B|| = n$, then $||A \supset B|| = n$;

if $||A|| = ||B|| = c$, then $||A \supset B|| \in \{n, c\}$;

if either $\{||A|| = c$ and $||B|| = i\}$ or $\{||A|| = n$ and $||B|| = c\}$, then $||A \supset B|| = c$;

$||A|| = n$ and $||B|| = i$, iff $||A \supset B|| = i$;

$||\Box A|| = n$ iff $||A|| = n$; $||\Box A|| = i$, iff $\{$either $||A|| = c$ or $||A|| = i\}$;

$||\Diamond A|| = i$, iff $||A|| = i$; $||\Diamond A|| = n$, iff $\{$either $||A|| = n$ or $||A|| = c\}$.

The formalisation and the proof of the meta-theorem of semantic completeness are the same as they were stated above.

3.4 Some peculiar properties of this logical system

First of all, it allows the use of the rule $A \Rightarrow \Box A$.

Besides, all derivable rule of inference of a classical propositional calculus are applicable to modalized formulas only. Some (at least some) direct rules of inference of a classical propositional calculus are also applicable to non-modalized formulas, for example: $A \vee B, \neg A \Rightarrow B$; but such indirect rules as rule of deduction:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

and rule reductio ad absurdum

$$\frac{\Gamma, A \Rightarrow B; \Gamma, A \Rightarrow \neg B}{\Gamma \Rightarrow \neg A}$$

are not applicable to non-modalized formulas in derivation.

However, so-called weakened rule of reductio ad absurdum

$$\frac{\Gamma, A \Rightarrow B; \Gamma, A \Rightarrow \neg B}{\Gamma \Rightarrow \Diamond \neg A}$$

is applicable to any formula in derivation.

4 Generalisation for quasimatrix logic

4.1 For logic S_{min}

LEMMA 4. suppose that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables; let A_i be a_i or $\neg a_i$, depending on whether b_i is t or f; let D' be D or $\neg D$ depending on whether D takes value t or f with truth-values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in every alternative interpretation, formed on the basis of some initial interpretation. Let D' be $D \vee \neg D$ depending on whether D takes value t under the truth assignment $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in some alternative interpretation formed on the basis of the initial interpretation, or it takes value f under the truth assignment $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in some alternative interpretation formed on the basis of the initial interpretation. Then $A_1, ..., A_n \Rightarrow D'$.

If in some alternative interpretations formula D takes value t and in some alternative interpretations it takes value f, then statement '$A_1, ..., A_n \Rightarrow D \vee \neg D$' may be substituted for the statement '$A_1, ..., A_n \Rightarrow D$ or $A_1, ..., A_n \Rightarrow \neg D$'.

PROOF. Lemma is proved by the use of recurrent mathematical induction.

Basis of induction. D does not contain any logical terms. Proof is obvious.

Assumption of induction. Proof holds for the formulas, containing k $(k \leq n)$ occurrences of logical terms.

Step of induction. Proof holds for the formulas containing $n+1$ occurrences of logical terms.

Case 1. $n+1$-th occurrence of the logical terms is the occurrence of the sign of negation. Formula D is $\neg B$.

Suppose formula D takes value t in all alternative interpretations, formed on the basis of some initial interpretation. Then B takes value f in all these alternative interpretations. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg B$.

Suppose formula D takes value f in all alternative interpretations, formed on the basis of some initial interpretation. Then B takes value t in all these alternative interpretations and by the assumption of induction $A_1, ..., A_n \Rightarrow B$. Then $A_1, ..., A_n \Rightarrow \neg\neg B$.

Under the third possibility $A_1, ..., A_n \Rightarrow \neg B \vee \neg\neg B$.

Case 2. $n+1$-th occurrence of the logical terms is the occurrence of the sign of necessity. Formula D is $\Box B$. Suppose B takes value f in all alternative interpretations, formed on the basis of some initial interpretation. Then by the assumption of induction $A_1, ..., A_n \Rightarrow \neg B$. Since $\neg B \supset \neg\Box B$ is a theorem scheme (contraposition of axiom scheme $\Box B \supset B$), then $A_1, ..., A_n \Rightarrow \neg\Box B$. If B takes value t in all or some alternative interpretations, then formula $\Box B$ takes value t in some alternative interpretations and in some other alternative interpretations it takes value f. Then it is obvious that $A_1, ..., A_n \Rightarrow \Box B \vee \neg\Box B$.

Case 3. $n+1$-th occurrence of the logical terms is the occurrence of the sign of possibility. Formula D is $\Diamond B$. Suppose B takes value t in all alternative interpretations, formed on the basis of some initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow B$. Since $B \supset \Diamond B$ is a theorem , $A_1, ..., A_n \Rightarrow \Diamond B$. If B takes value f in all or some alternative interpretations, then formula $\Diamond B$ takes value t in some alternative interpretations and it takes value f in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow \Diamond B \vee \neg\Diamond B$.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$. If formula D under above-mentioned truth-assignments of its variables takes value t in some alternative interpretations and in some other alternative interpretations it takes value f, then D' is $(B \supset C) \vee \neg(B \supset C)$. The entailment is obvious. If D takes value f, then D' is $\neg(B \supset C)$.

It is possible if in every alternative interpretation formula B takes value t and formula C takes value f. By the assumption of induction for every alternative interpretation holds that $A_1, ..., A_n \Rightarrow B$ and $A_1, ..., A_n \Rightarrow \neg C$. Consequently $A_1, ..., A_n \Rightarrow \neg(B \supset C)$. Let's take into consideration the last case, then D takes value t in every alternative interpretation. It means that in every alternative interpretation formula B takes value f or formula C takes value t. Hence by the assumption of induction,

$$A_1, ..., A_n \Rightarrow \neg B$$
or
$$A_1, ..., A_n \Rightarrow C.$$

Analyzing all possible cases we conclude: $A_1, ..., A_n \Rightarrow (B \supset C)$. $\square$

4.2 For logic S_r

LEMMA 5. Suppose that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are values of these variables; let A_i be $\Box a_i$, $\Diamond a_i \& \Diamond \neg a_i$, $\neg \Diamond a_i$, depending on whether b_i is n, c, or i. Let D' be $\Box D$, $\Diamond D \& \Diamond \neg D$ or $\neg \Diamond D$, depending on whether D takes value n, c, or i with values $b_1, ..., b_n$ variables $a_1, ..., a_n$ in all alternative interpretations, formed on the basis of some initial interpretation; suppose D' is $\Box D \vee (\Diamond D \& \Diamond \neg D)$, $\Box D \vee \neg \Diamond D$, $(\Diamond D \& \Diamond \neg D) \vee \neg \Diamond D$, $(\Box D \vee (\Diamond D \& \Diamond \neg D)) \vee \neg \Diamond D$, depending on whether D takes, respectively, value n in some alternative interpretations and in some other alternative interpretations it takes value c; D takes value n in some alternative interpretations and in some others it takes value i; D takes value c in some alternative interpretations and in some others it takes value i; D takes value n in some alternative interpretations or it takes value c in some other alternative interpretations, or it takes value i in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow D'$.

If D' is $\Box D_i \vee (\Diamond D_i \& \Diamond \neg D_i)$, statement '$A_1, ..., A_n \Rightarrow D'$' may be substituted for '$A_1, ..., A_n \Rightarrow \Box D_i$ or $A_1, ..., A_n \Rightarrow \Diamond D_i \& \Diamond \neg D_i$'. The substitution of the same kind is possible in case of other values in different alternative interpretations. I.e, logical entailment is based on alternative interpretations formed on the basis of some initial interpretation. For example, if formula takes value n in every alternative interpretation, then the following holds for these alternative interpretations '$A_1, ..., A_n \Rightarrow \Box D_i$ or $A_1, ..., A_n \Rightarrow \Box D_i$, or $A_1, ..., A_n \Rightarrow \Box D_i$'. Hence $A_1, ..., A_n \Rightarrow \Box D_i$. Note that if there is no any ambiguity the only alternative interpretation that is possible is the initial one. In this case $A_1, ..., A_n \Rightarrow \Box D_i$ also holds. The same holds for the other values.

Proof. Lemma is proved by recurrent mathematical induction on the number of occurrences of logical terms in formula D.

Step of induction.

Case 1. Formula D is $\neg B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$. $\neg\Diamond B \supset \Box\neg B$ is a theorem scheme. (Using theorem scheme $\neg\Box\neg A \supset \Diamond A$.) Then $A_1, ..., A_n \Rightarrow \Box\neg B$.

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value n in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond\neg B$. Here we use the axiom scheme $\Diamond A \supset \neg\Box\neg A$ and the rule of substitution of $\neg\neg A$ for A and vice versa.

Suppose D takes value c in every alternative interpretation formed on the basis of some initial interpretation. Then B also takes value c in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$. Hence $A_1, ..., A_n \Rightarrow (\Diamond\neg B \& \Diamond\neg\neg B)$.

Suppose D takes value n in some alternative interpretations and it takes value c in some others. By the assumption of induction: $A_1, ..., A_n \Rightarrow \neg\Diamond B$ or $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$. Since in the first case $A_1, ..., A_n \Rightarrow \Box\neg B$ and in the second one $A_1, ..., A_n \Rightarrow (\Diamond\neg B \& \Diamond\neg\neg B)$, the following holds: $A_1, ..., A_n \Rightarrow \Box\neg B \vee (\Diamond\neg B \& \Diamond\neg\neg B)$.

For other possible cases proof is analogous.

Case 2. Formula D is $\Box B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B also takes value n in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \Box\Box B$. (Using axiom scheme $\Box A \supset \Box\Box A$.)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation, or it takes value c in every alternative interpretation, or it takes value i in some alternative interpretation and it takes value c in some another alternative interpretation. Under the last possibility by the assumption of induction

$$A_1, ..., A_n \Rightarrow \neg\Diamond B$$
or
$$A_1, ..., A_n \Rightarrow (\Diamond B \& \Diamond\neg B).$$

In both cases $A_1, ..., A_n \Rightarrow \neg\Diamond\Box B$. (In the first case we use axioms schemes $\Diamond\Box A \supset \Box A$ and $\Box A \supset \Diamond A$, and in second one – $\Diamond\Box A \supset \Box A$ and $\Diamond A \supset \neg\Box\neg A$.) Formula D can not take value c.

If formula D takes different truth values in different alternative interpretations the proof may be concluded from the above-analyzed cases.

Case 3. Formula D is $\Diamond B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value n in every alternative interpretation, or it takes value c in every alternative interpretation, or it takes value n in some alternative interpretation and it takes value c in another alternative interpretation. Under the last possibility by the assumption of induction

$$A_1, ..., A_n \Rightarrow \Box B$$
or
$$A_1, ..., A_n \Rightarrow (\Diamond B \& \Diamond\neg B).$$

In both cases $A_1, ..., A_n \Rightarrow \Box\Diamond B$. (In the first case we use axioms schemes $\Box A \supset \Diamond A$ and $\Diamond A \supset \Box\Diamond A$, and in the second case we need only the last axiom)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B$. (Using the axiom scheme $\Diamond\Diamond A \supset \Diamond A$.) Formula D can not take value c. If formula D takes different truth values in different alternative interpretations the proof may be concluded from the above-analyzed cases.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value i in every alternative interpretation or C takes value n in every alternative interpretation. By the assumption of induction for every alternative interpretation holds: $A_1, ..., A_n \Rightarrow \neg\Diamond B$ or $A_1, ..., A_n \Rightarrow \Box C$. Hence: $A_1, ..., A_n \Rightarrow \Box(B \supset C)$. (Using axiom schemes $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$.)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value n in every alternative interpretation and C takes value i in every alternative interpretation. By the assumption of induction for every alternative interpretation holds: $A_1, ..., A_n \Rightarrow \Box B$ и $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. (Using axiom schemes $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.)

Suppose D takes value c in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value n and C takes value c in every alternative interpretation or B takes value c and C takes value i in every alternative interpretation. In the first case $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow \Diamond C \& \Diamond\neg C$. Then we have to prove: $A_1, ..., A_n \Rightarrow \Diamond(B \supset C) \& \Diamond\neg(B \supset C)$.

$A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using theorem scheme $\Diamond B \supset \Diamond(A \supset B)$). $A_1, ..., A_n \Rightarrow \Diamond\neg(B \supset C)$ (using axiom schemes $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and $\neg\Box\neg A \supset \Diamond A$, and rule of substitution of $\neg\neg A$ for A and vice versa). In second case $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$, and $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Then $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using axiom scheme $\Diamond\neg B \supset \Diamond(A \supset B)$). $A_1, ..., A_n \Rightarrow \Diamond\neg(B \supset C)$ (using axiom schemes $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ and $\neg\Box\neg A \supset \Diamond A$).

Suppose D takes value n in some alternative interpretation formed on the basis of some initial interpretation and it takes value c in another interpretation. Then we have to prove: $A_1, ..., A_n \Rightarrow$

$\Diamond(B \supset C)\&\Diamond\neg(B \supset C)$ or $A_1, ..., A_n \Rightarrow \Box(B \supset C)$, or the equivalent statement $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$. This case is possible if both B and C takes value c in all alternative interpretations. By the assumption of induction $A_1, ..., A_n \Rightarrow \Diamond B\&\Diamond\neg B$ and $A_1, ..., A_n \Rightarrow \Diamond C\&\Diamond\neg C$. Then $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using axiom scheme $\Diamond B \supset \Diamond(A \supset B)$).

The proof of other possibilities may be concluded from the above-analyzed cases. □

METATHEOREM 1. *If formula D is universally satisfiable then it is provable.*

Since for every truth-assignment of the variables holds $A_1, ..., A_n \Rightarrow \Box D$, then the following holds:

1. $A_1, ..., A_{n-1}, \Box a_n \Rightarrow \Box D$,

2. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow \Box D$,

3. $A_1, ..., A_{n-1}, \Diamond a_n\&\Diamond\neg a_n \Rightarrow \Box D$.

   Hence:

4. $A_1, ..., A_{n-1}, \Diamond a_n, \neg\Diamond\neg a_n \Rightarrow \Box D$, from 1,

5. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow \Box D$, from 2,

6. $A_1, ..., A_{n-1}, \Diamond a_n, \Diamond\neg a_n \Rightarrow \Box D$, from 3.

7. $A_1, ..., A_{n-1}, \Diamond a_n \Rightarrow \Box D$, from 4, 6,

8. $A_1, ..., A_{n-1} \Rightarrow \Box D$, from 5, 7. etc.

As $\Box D$ entails D, D is provable.

REMARK 1. Since formula can take one of the seven values (n, c, i, n/c, n/i, c/i, $n/c/i$), the problem arises to construct 7-valued logic with this values (lets sign them with 1, 2, 3, 4, 5, 6, 7) and compare it with S_r.

4.3 For logic Sa-

LEMMA 6. Suppose D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables; let A_i be $\Box a_i$, $a_i \& \Diamond \neg a_i$, $\neg \Diamond a_i$, $\neg a_i \& \Diamond a_i$, depending on whether b_i is t^n, t^c, f^i or f^c. Let D' be $\Box D$, $D \& \Diamond \neg D$, $\neg \Diamond D$ or $\neg D \& \Diamond D$, depending on whether D takes value t^n, t^c, f^i or f^c with values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in all alternative interpretations formed on the basis of some initial interpretation. Suppose D' is $\Box D \vee (D \& \Diamond \neg D)$, $\Box D \vee \neg \Diamond D$, $(D \& \Diamond \neg D) \vee \neg \Diamond D$, $(\Box D \vee (D \& \Diamond \neg D)) \vee \neg \Diamond D$ and so on, depending on whether D takes respectively value t^n in some alternative interpretations and in some other alternative interpretations it takes value t^c; D takes value t^n in some alternative interpretations and in some others it takes value f^i; D takes value t^c in some alternative interpretations and in some others it takes value f^i; D takes value t^n in some alternative interpretations or it takes value t^c in some other alternative interpretations, or it takes value f^i in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow D'$.

PROOF. Lemma is proved by recurrent mathematical induction on the number of occurrence of logical terms in formula D.

Step of induction.

Case 1. Formula D is $\neg B$.

Suppose D takes value t^n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^i in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg \Diamond B$. $\neg \Diamond B \supset \Box \neg B$ is a theorem scheme. (Using theorem scheme $\neg \Box \neg A \supset \Diamond A$.) Then $A_1, ..., A_n \Rightarrow \Box \neg B$.

Suppose D takes value f^i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \neg \Diamond \neg B$. Here we use an axiom scheme $\Diamond A \supset \neg \Box \neg A$ and rule of substitution of $\neg \neg A$ for A and vice versa.

Suppose D takes value t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^c in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$. Hence $A_1, ..., A_n \Rightarrow \neg B \& \Diamond \neg\neg B$.

Suppose D takes value f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^c too in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow B \& \Diamond \neg B$. Hence $A_1, ..., A_n \Rightarrow \neg\neg B \& \Diamond \neg B$.

Suppose D takes value t^n in some alternative interpretations formed on the basis of some initial interpretation and it takes value t^c in some other interpretations. By the assumption of induction B takes value f^i in some alternative interpretations and it takes value f^c in other alternative interpretations. Then $A_1, ..., A_n \Rightarrow \neg \Diamond B$ or $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$.

Since in the first case $A_1, ..., A_n \Rightarrow \Box \neg B$ and in the second $A_1, ..., A_n \Rightarrow \neg B \& \Diamond \neg\neg B$, the following holds: $A_1, ..., A_n \Rightarrow \Box \neg B \vee (\neg B \& \Diamond \neg\neg B)$.

For other possible cases proof is analogous.

Case 2. Formula D is $\Box B$.

Suppose D takes value t^n or t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then we have to prove: $A_1, ..., A_n \Rightarrow \Box\Box B \vee (\Box B \& \Diamond \neg \Box B)$.

$\Box\Box B \vee (\Box B \& \Diamond \neg \Box B) \Leftrightarrow (\Box\Box B \vee \Box B) \& (\Box\Box B \vee \Diamond \neg \Box B)$.

$(\Box\Box B \vee \Box B) \& (\Box\Box B \vee \Diamond \neg \Box B) \Leftrightarrow (\Box\Box B \vee \Box B) \& (\Box\Box B \vee \neg \Box\Box B)$.

$(\Box\Box B \vee \Box B) \& (\Box\Box B \vee \neg \Box\Box B) \Leftrightarrow \Box B$.

Proof is completed. ($\Leftrightarrow$ is a sign for metalanguage equivalence).

Suppose D takes value f^i or f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^c or f^i, or f^c in every alternative interpretation formed on the basis of this initial interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \neg \Diamond \Box B \vee (\neg \Box B \& \Diamond \Box B)$. That is, we have to prove: $A_1, ..., A_n \Rightarrow \neg \Box B$.

In the first case by the assumption of induction $A_1, ..., A_n \Rightarrow B \& \Diamond \neg B$. The proof is evident.

In the second case $A_1, ..., A_n \Rightarrow \neg\Diamond B$. $\neg\Diamond B \Rightarrow \neg\Box B$. (Using axiom schemes $\neg\Box\neg A \supset \Diamond A$ and $\Box A \supset A$.) The statement is proved.

In the third case $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$. $\neg B \Rightarrow \neg\Box B$. The statement is proved.

Case 3. Formula D is $\Diamond B$.

Suppose D takes value t^n or t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n or t^c, or f^c in every alternative interpretation formed on the basis of this initial interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \Box\Diamond B \vee (\Diamond B \wedge \Diamond\neg\Diamond B)$. That is we have to prove: $A_1, ..., A_n \Rightarrow \Diamond B$. By the assumption of induction in every of three cases $A_1, ..., A_n \Rightarrow \Diamond B$.

Suppose D takes value f^i or value f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^i in every alternative interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B)$. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$.

$\neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B) \Leftrightarrow \neg\Diamond B$

So $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B)$ is proved.

Cases when formula D takes different values in different alternative interpretations may be reduced to the above-analyzed cases.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$.

Suppose formula D takes value t^n in every alternative interpretation. It is possible if either B takes value f^i or C takes value t^n. We have to prove: $A_1, ..., A_n \Rightarrow \Box(B \supset C)$. The statement may be easily proved by axiom schemes $\neg\Diamond A \supset \Box(A \supset B)$, $\Box A \supset \Box(A \supset B)$.

Suppose formula D takes value f^i in every alternative interpretation. Then B takes value t^n and C takes value f^i. We have to prove: $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Hence, $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. (Using axiom scheme $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.)

Suppose formula D takes value t^c in every alternative interpretation. It is possible if both B and C takes value t^c in every alternative interpretation, or if B takes value t^n and C takes value t^c, or if B takes value f^c and C takes one of the three values: t^c or f^i or f^c.

We have to prove: $A_1, ..., A_n \Rightarrow (B \supset C)\&\Diamond\neg(B \supset C)$. Under the first condition $A_1, ..., A_n \Rightarrow B\&\Diamond\neg B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$. $C \Rightarrow B \supset C$. $B \Rightarrow \Diamond B$. $\Diamond\neg C \Rightarrow \neg\Box C$. $\Diamond B\&\neg\Box C \Rightarrow \Diamond\neg(B \supset C)$. (Using axiom schemes $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$, $\neg\Box\neg A \supset \Diamond A$.)

Under the second condition $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$. The proof is the same as in the previous case.

Under the third condition $A_1, ..., A_n \Rightarrow \neg B\&\Diamond B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$ or $A_1, ..., A_n \Rightarrow \neg\Diamond C$, or $A_1, ..., A_n \Rightarrow \neg C\&\Diamond C$. In any case $A_1, ..., A_n \Rightarrow \neg\Box C$. The proof is completed. $\Box$

Cases when formula D takes different values in different alternative interpretations may be reduced to the above-analyzed cases.

METATHEOREM 2. *If formula D is universally satisfiable then it is provable.*

(Since for every truth-assignment of the variables holds $A_1, ..., A_n \Rightarrow \Box D$ or $A_1, ..., A_n \Rightarrow (D\&\Diamond\neg D)$ then the following holds: $A_1, ..., A_n \Rightarrow D$.)

1. $A_1, ..., A_{n-1}, \Box a_n \Rightarrow D$,
2. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow D$,
3. $A_1, ..., A_{n-1}, a_n\&\Diamond\neg a_n \Rightarrow D$,
4. $A_1, ..., A_{n-1}, \neg a_n\&\Diamond a_n \Rightarrow D$,
   Hence
5. $A_1, ..., A_{n-1}, \neg\Diamond\neg a_n \Rightarrow D$, from 1,
6. $A_1, ..., A_{n-1}, \neg a_n, \neg\Diamond a_n \Rightarrow D$, from 2,
7. $A_1, ..., A_{n-1}, a_n, \Diamond\neg a_n \Rightarrow D$, from 3,
8. $A_1, ..., A_{n-1}, \neg a_n, \Diamond a_n \Rightarrow D$, from 4,
   And then:
9. $A_1, ..., A_{n-1}, a_n \Rightarrow D$, from 5, 7,
10. $A_1, ..., A_{n-1}, \neg a_n \Rightarrow D$, from 6, 8,
11. $A_1, ..., A_{n-1} \Rightarrow D$, from 9, 10, and so forth.

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The concept of ‘Translation’: history and theory

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abstract. This article deals with the problem of translations. It covers the history of translation in linguistics and analyzes peculiarities and role of translation in logic. Moreover, the article contains typical examples of embedding operations in terms of different logical theories.

Keywords: logic, translation, embedding, embedment, operation, language, calculi, theory

1 ‘Theory of translation’ in linguistics

The French poet and philosopher, humanist Etienne Dolet [18, p. 6] in the XVI century was one of the first, who was trying to formulate a theory of translation in order to impart scientific justification into this kind of activity. According to his point of view, the true translation must meet the following criteria: perfect understanding of the content of the original text and the author’s intentions; mastery of languages, which are involved in the translation process; inappropriateness of literal translation (in order to preserve the authentic atmosphere of the original); preserving the style of the original text, etc.

Later, another researcher T. Sevori [18, pp. 13–14] made a list of requirements for the various authors translations. Outlining of some of his statements will be enough to get his point of view on translation process: 1) the translation must convey the source words, 2) the translation must convey the source ideas, 3) the translation must be read like the original, 4) the translation must be read like a translation, etc. It is important to note that different scientists at different periods of time sometimes demanded from translation totally antithetic requirements. Hereafter, many translators, serious writers, and, finally, linguists made their own lists of requirements for 'translation' and then provide them with the corresponding theoretical justification. Here we are not going to discuss such research, carried out within this framework. Our main goal is to discuss the 'linguistic theory of translation', as so far scientific, well substantiated discipline.

The foundations of scientific translation theory were developed in the mid-twentieth century. That was the time of close attention of linguisticians and linguists to this problem. Some philosophers cast doubts on the possibility of translation in general — V. Humboldt, in particular [11]. According to him, every translation is an attempt to solve unsolvable task because of the dependence on the personal characteristics of the individual translator and his attitude to the text. Similar views served as the forerunners of the 'theory of untranslatability'.

The doubts on the possibility of the translation studies by the methods of linguistics were dispelled when this phenomenon became known as a special kind of verbal activity. This kind of activity was admitted to be the one, in which the units of the target language are selected depending on the specific language units of the source text, but not as a result of individual translation creativity. Here, apparently, there was a final demarcation of this area on the direct field of translation activities and the theory of translation.

J. Vinai and J. Darbelnet's attempt to subject different languages to the comparative analysis contributed a lot to the development of the linguistic theory of translation [29]. The analysis helps to detect units from different languages, which can be used in translation interchangeably. These words should a priori carry the same meanings in their two language systems or should come up as equivalents at the end of the translation.

Another linguist R. Jakobson [12] introduced an idea that the theory of translation plays an important role in other sciences, particularly in different branches of linguistics. He offered to define different types of translation: intralingual, interlingual and intersemiotic, in which one system of signs transforms into another. He supposed every transformation which carries the original meaning to be an adequate translation. This idea of R. Jakobson is closely correlated with the role of translation in logic.

G. Mounin in [21] paid special attention to the semantic structures of languages and claimed to square up to their differences. They occur because semantic discrepancies impose certain translation restrictions, for instance, make it impossible to render the original meaning fully.

It is also important to mention the merits of Russian scientists I.I. Revsin and V.U. Rosentsveig to the linguistic translation theory. In [26] they pointed out that the theory of translation should be not a prescriptive (i.e. the one that a priori formulates translation demands), but a descriptive — the one to describe the objective reality. It is the description that produces standards and regulatory guidelines of translation. They also highlighted the use of the deductive approach, which extends the use of general linguistic concepts in translation process.

I.I. Revsin and V.U. Rosentsveig defined two methods to transform source text to translated one: a) the direct substitution of the units of the source language into the units of another, b) interpretation. Last was meant to comprehend the reality at first, described by the original language, and then to descript it by the means of the target language.

J. Catford in [3] presumed that the central problem of the translation theory is to render the meaning of original and translated statements adequately close. In such a way he raised the question of texts equivalence in translation. He supposed that the original language meaning is replaced with the meaning of the target one, therefore the equivalence depends on the accuracy of such replacement.

V.N. Komissarov created an integrated theoretical conception in [17]. There he summarized different aspects of the linguistic analysis of translation, classified the research datum, including some mentioned above. Special attention was eftsoons paid to the problem of equivalence.

The issues stated above are just several fragments of the huge linguistic mosaics of translation theory problematic field. On the assumption of asserted, it is possible to point out two major inferences. These inferences will be the requirements basis for defining the term 'translation" and for comparing other definitions with the worked out one later. First of all, any translation should keep the meaning of both the original and target texts. In other words, the problem of invariance should be taking into account while crossing the boundaries of two languages. Secondly, the target text should be theoretically equivalent to the original one. The logical sequence and inter-originating of these two principles can be clearly observed.

2 'Translation' in logic

The area of interest for this research includes particular use of translation method — more specifically, transfers between logical calculi. Hereunder, the concept of translation would require clarification. This concept stays in contrast to the linguistic requirements for translation, which state saving of semantic units as one of the key conditions. It is necessary to preserve the verity for compliance of translation in logic. That means, logical truths of one language should be translated into logical truths of another.

The consequence of the R. Yakobson's requirement about saving of semantic units is that 1) the true statement translation result is a true statement. As it was already mentioned, the logical verity is important in logic. Considering that, the adaptation in logic angle of the requirement 1) formulated above is the claim that 1)$'$ the true logic statement translation result is a true logic statement. Taking into consideration this requirement, the definition of translation will be:

DEFINITION 1 (DF1). The translation of calculus C_1 into the calculus C_2 meant to be such a mapping of φ set of all L_1-formulas of calculus C_1 into the set of all L_2-formulas of calculus C_2, that for every L_1-formula A the following condition holds:

$$\textit{if } \vdash_{C_1} A, \textit{ then } \vdash_{C_2} \varphi(A).$$

However, any calculus will be transferred to any other calculus with this understanding of translation. That leads to questioning the retention of the source statement meaning — plus the fact, that logically true statements are not always equivalent in logical calculi.

Besides, it remains unknown, how the translation result corresponds to the original. According to J. Catford, it is essential to estimate semantic affinity between the statements in the original and in the translation in order to verify their equivalence. In addition, this definition does not take into consideration the content of the statements. Therefore, it does not lead to source reconstitution by the translation, that is, in principle, impossible as it was indicated by G. Mounin.

Then let's introduce another requirement:

DEFINITION 2 (DF2). The translation of calculus C_1 into the calculus C_2 meant to be such a mapping of φ set of all L_1-formulas of calculus C_1 into the set of all L_2-formulas of calculus C_2, that for every L_1-formula A the following condition holds:

$$\vdash_{C_1} A, \textit{ if and only if } \vdash_{C_2} \varphi(A).$$

In contrast to the first definition, there is an opportunity to check the logical validity of the first calculus statement, considering the assumption of the logical validity of its image in the second calculus statement. Now it is possible to correlate the translation with the original. But most of the former lacks last even in this definition.

To avoid them, let's add to Df2 the following condition for avoiding those lacks — φ is a recursive function. Then the new definition will be Df3.

But there are a number of drawbacks even in such way. According to R. Epstein [6, p. 291], for example, this definition does not guarantee the safety of the produced formulas structures. That means the impossibility of full source content reproduction on free analogy with Mounin's remark. Then it is necessary to take into account the structure of the formulas, that, in fact, is the requirement of inductive definition? (let‘s name this definition Df4).

Is it enough to give a good definition of the translation? Apparently, not. The question remains, whether the equivalence of the original and the translation result observes?

It is necessary to work out more strict criteria for what is called 'translation'. Below such definitions proposed directly by logics will be considered.

3 Definitions of embedding operations

Let's analyze and compare the most important, in our opinion, definitions, suggested by V.A. Smirnov, R. Wojcicki and R. Epstein.

In [28] V.A. Smirnov defined the translations between theories. However, his definition will also be relevant for the calculi with the appropriate modification.

Suppose, that T_1 and T_2 are theories, formulated accordingly in languages L_1 and L_2 with the corresponding logics. Suppose, that φ is a recursive function that matches formulas in language L_1 with formulas in language L_2 for any L_1-formula A. The function meant to be called translation of theory T_1 to T_2, if the following condition holds: if $A \in \mathrm{T}_1$, then $\varphi(\mathrm{A}) \in \mathrm{T}_2$. If the following additional condition holds: if $\varphi(\mathrm{A}) \in \mathrm{T}_2$, then $A \in \mathrm{T}_1$, then the recursive function φ would be called an embedding operation of the theory T_1 to the theory T_2. Theory T_1 could be embedded to the theory T_2, if and only if there is a recursive function, which embeds T_1 to T_2. Of course, if there is a translation — there is not necessarily the case of embedment. Later we will talk more about the 'embedment' since we are deeply interested in cases that satisfy both Smirnov's conditions at once.

V. A. Smirnov's definition actually coincides with the above Df3.

R. Wòjcicki offered a different definition [30].

Mapping φ from sentential language L_1 to sentential language L_2, which have the same set of propositional variables, is called the embedment if and only if the following two conditions hold:

1. There is a formula $\phi(p_0)$ from one propositional variable p_0 in L_2, such that for every propositional variable p, $\varphi(p) = \phi(p_0)$.

2. For each logical connective r_i in L_1 there is a formula ϕ_i in L_2, such that for all $\alpha_1, \ldots, \alpha_k$ in L_1, k is the arity r_i,

$$\varphi(r_i(\alpha_1, \ldots, \alpha_k)) = \phi_i(p_1/\varphi(\alpha_1), \ldots, p_k/\varphi(\alpha_k)).$$

Then, the definition of embedment is stated for propositional calculi and theories.

Suppose that $C_1 = (L_1, C_1)$, $C_2 = (L_2, C_2)$ are some propositional calculi, and $T_1 = (L_1, T_1)$, $T_2 = (L_2, X_2)$ are some theories, then φ is the embedding operation from L_1 to L_2, if for every $T \subseteq L_1$ and for every $\alpha \in L_1$ the following holds:

$$\alpha \in \mathrm{C}_1(T) \textit{ if and only if } \varphi(\alpha) \in \mathrm{C}_2(\varphi(T))(\varphi(T_1) = T_2).$$

Epstein offered a similar definition (ref. [9, pp. 290–291]), but with some differences, which will be defined in detail below.

At first, he formulated the definition of a mapping from one logic to another. This helps to preserve the relation of deductivity.

Mapping of the propositional logic L to the propositional logic M, which preserves the relation of deductivity, is a mapping φ from language L_L to language L_M (L_L и L_M are the languages of logics L and M, respectively), such that for every formula A the following condition holds:

$$\vdash_L A \textit{ if and only if } \vdash_M \varphi(A).$$

Mapping is a translation, if for any G and A the following condition holds:

$$G \vdash_L A \textit{ if and only if } \varphi(G) \vdash_M \varphi(A), \textit{ where } G \textit{ is the set of formulas, and } \varphi(G) = \{\varphi(A) : A \in G\}.$$

According to Epstein, such definition of embedment does not cause the preservation of target language structure. This appears to be valid, especially while discussing such examples, as Glivenko's translation of classical propositional logic into the intuitionistic logic [9], which matches the above specified definition.

Epstein then introduced the concept of grammatical mapping:

Mapping φ of propositional language $L_{1\supset\neg}$ to any propositional language L_2 is called grammatical, if there are schemes λ, φ, ψ in language L_2, such that

$$\begin{gathered} p^* = \lambda(p), \\ (\neg A)^* = \varphi(A^*), \\ (A \supset B)^* = \psi(A^*, B^*), \end{gathered}$$

whereas it is implied, that in these languages, the set of propositional variables is the same.

Grammatical mapping is called homophonic if each connective maps into itself. Grammatical embedment is a grammatical mapping, which comes out as an embedment.

R. Wòjcicki and R. Epstein's definitions meet the one, which was marked above as Df4.

4 Comparison of definitions

As is clear from aforementioned definitions, all of them have specific differences. V.A. Smirnov provides the broadest definition. Glivenko's contribution resulted in translation of the classical propositional logic into the intuitionistic one. His translation meets the definition of V. A. Smirnov, but falls outside the scope of R. Wòjcicki and R. Epstein's meaning of embedment.

Let us denote the differences in the definitions of Wòjcicki and Epstein. There is a need to reformulate them in a similar style and then compare.

Suppose the languages L' and L'' are given with one and the same set of propositional variables, $\supset$ and $\neg$ are the logical connectives of L' language, φ is the embedding operation from L' language to L'', and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are some formulas of L'' language.

Then the definition of Wòjcicki can be represented in the following form:

- $\varphi(p) = [p_0/p]\boldsymbol{A}$, where $[p_0/p]\boldsymbol{A}$ is the result of substitution p instead of p_0 into the formula $\boldsymbol{A}$, which does not contain occurrences of propositional variables, other than p_0,
- $\varphi(A \supset B) = [p_1/\varphi(A), p_2/\varphi(B)]\boldsymbol{B}$, where $[p_1/\varphi(A), p_2/\varphi(B)]\boldsymbol{B}$ is the result of substitution $\varphi(A)$ instead of p_1 and $\varphi(B)$ instead of p_2 into the formula $\boldsymbol{B}$,
- $\varphi(\neg A) = [p_1/\varphi(A)]\boldsymbol{C}$, where $[p_1/\varphi(A)]\boldsymbol{C}$ is the result of substitution $\varphi(A)$ instead of p_1 into the formula $\boldsymbol{C}$, which does not contain occurrences of propositional variables, other than p_1.

Epstein claims the following:

- $\varphi(p) = [p_0/p]\boldsymbol{A}$, where $[p_0/p]\boldsymbol{A}$ is the result of the substitution p instead of p_0 into the formula $\boldsymbol{A}$,
- $\varphi(A \supset B) = [p_1/\varphi(A), p_2/\varphi(B)]\boldsymbol{B}$, where $[p_1/\varphi(A), p_2/\varphi(B)]\boldsymbol{B}$ is the result of substitution $\varphi(A)$ instead of p_1 into the formula $\boldsymbol{B}$,
- $\varphi(\neg A) = [p_1/\varphi(A)]\boldsymbol{C}$, where $[p_1/\varphi(A)]\boldsymbol{C}$ is the result of substitution $\varphi(A)$ instead of p_1 into the formula $\boldsymbol{C}$, which does not contain occurrences of propositional variables, other than p_1.

So the only varying cases are those, which hold the scope of the embedding operation within propositional variables. Otherwise, R. Wòjcicki and P. Epstein's definitions stay matching.

But wherein does the complexity of embedment universal definition as a means for comparison and study of logical systems lie? As referred to M.N. Rybakov and A.V. Chagrov in [4], it makes sense to impose additional conditions (other than those, that were put forward in the above definition), depending on the purpose of a particular embedment, since it is often necessary to consider the contents of the formulas. Otherwise, it is impossible to represent adequately the formulas of one logic by the means of another logic. In other words, in any embedment could be a list of requirements for embedding operation.

The definition proposed by V.A. Smirnov would be the basis for this research on different embedments of logical calculi. This was motivated by simplicity and convenience of his definition. But our embedding operations are also true within R. Wòjcicki and R. Epstein's theories, as it will become clear from their construction.

5 Reasons for embedding operations application in logic. Philosophical and technical aspects

There are several reasons for the use of embedding operations in logic. Here we can highlight technical and philosophical aspects.

As it was mentioned above, philosophically, embedment helps to map one theory by another theory terms. The ability of comparing theories, formulated in different languages, becomes real with such a phenomena, but stands in opposition to Firebrand's idea of 'incommensurability of theories'. According to this idea terms of one theory cannot be expressed in terms of another one, as they themselves have different meanings. The embedding operations method (if we can call it a method) removes the 'problem of understanding'. If we embed non-interpreted calculus into calculus, which has some semantic meaning, the interpretation of first embedment formulas becomes possible.

This raises the problem of negation in language: embedment of containing negation language to a different positive language or its own positive part we get an opportunity to speak about the first language facts using only affirmative sentences, i.e. without saying any 'no'.

Speaking about the technical aspect we should note the problem of decidability. Hereby, embedment of any calculus into a decidable one results into solving the problem of decidability of the first calculus. The problem of languages relative insolvability is also observed in terms of this problem. Embedment of calculi into their own fragments and other calculi fragments decreases the number of connectives, which are necessary to express formulas in different languages. This is again very important for understanding of the original calculi. V.M. Popov got one of the most eloquent and unexpected results in this area, when he embedded classical propositional logic into its implicative fragment and implicative fragment of intuitionistic propositional logic. We think that this idea demands some serious deliberation, as it is not quite clear.

Besides, as it was shown in V.M. Popov's research in [22], use of the embedding operations helps to prove fragment severability in calculi. The characteristics, mentioned here, can be added to much broader list of application areas than discussed here.

6 The history of the concept of 'embedment'. Specific embedding operations in classical and intuitionistic propositional logics

Besides the above mentioned Smirnov, Wòjcicki and Epstein, attempts to determine and organize embedding operations were also taken by N.A. Shanin [27], who, according to V.A. Smirnov (ref. [28, p. 120]), had first coined the term, D. Prawitz and P. Mamnos [25], W.A. Carnielli and M.L. D'Ottaviano [2]. Based upon the last work of Carnielli and D'Ottaviano, A.S. Karpenko concluded that the application of embedding operations is a key tendency in the development of contemporary logic [13].

The term of embedding operation was first time introduced, according to V.A. Smirnov [28, p. 120], by A. N. Kolmogorov in 1925 [16] while embedding classical logic into intuitionistic. Particular attention is worth paying to the problem of relation between classical and intuitionistic logics towards the embedding operations after many leading scientists who made it a point.

For that let's specify after [15] the calculi *PC* (classical propositional calculus) and *Int* (intuitionistic propositional calculus).

Language $L_{\wedge\vee\supset\neg}$ of these calculi is the conventionally determined propositional language with a set of propositional variables $\{p_1, p_2, p_3, \dots\}$.

Calculi *PC* and *Int* are the calculi of Hilbert type with the conventionally determined concept of proof. The set of deduction rules for each of these calculi has the only rule: $\mathrm{A}, \mathrm{A} \supset \mathrm{B}/\mathrm{B}$. Therefore, it is sufficient to define the set of its axioms in order to specify any of these calculi.

Calculus *PC*. The set of all axioms of calculus *PC* is the set of all formulas, each of which is stated in at least one of the following types:

1. $A \supset (B \supset A)$,
2. $A \supset (B \supset C) \supset ((A \supset B) \supset (A \supset C))$,
3. $(A \wedge B) \supset A$,
4. $(A \wedge B) \supset B$,
5. $A \supset (B \supset (A \wedge B))$,
6. $A \supset (A \vee B)$,
7. $B \supset (A \vee B)$,

8. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$,
9. $(A \supset B) \supset ((A \supset (\neg B)) \supset (\neg A))$,
10. $(\neg(\neg A)) \supset A$.

Calculus *Int*. The set of all axioms of *Int* is the union of the set of all formulas, each of which is a formula of at least one of the above mentioned 1–8 types with the set of all formulas of the type:

$9'$ $(\neg A) \supset (A \supset B)$.

Glivenko in 1929 suggested embedding operation for classical and intuitionistic logics, which lies within the language L — the conventionally determined propositional language with a set of logical connectives $\{\wedge, \vee, \supset, \neg\}$. This embedding operation associates each L-formula A with L-formula $\neg(\neg A)$. Precisely, Glivenko proved the following theorem, which let's call T1 for convenience of reference.

THEOREM 1 (T1). *$G \vdash_{PC} A$ if and only if $\neg(\neg G) \vdash_{PC} \neg(\neg A)$.*

V. Popov noted an interesting fact about Glivenko's embedding operation to fail for the predicative versions of intuitionistic and classical logics.

In 1933 Gödel showed [10] that *Int* could be considered as the extension of classical propositional logic, formulated in the language $L_{\wedge\neg}$ within the meaning of the following theorem (let's call it T2).

THEOREM 2 (T2). *For $L_{\wedge\neg}$-formula A it is true that $\vdash_{PC_{\wedge\neg}} A$ if and only if $\vdash_{INT} A$.*

PROOF. The proof of this theorem right to left is obvious, because the set of all *Int* theorems is included into the set of all *PC* theorems. Let's prove that if $\vdash_{PC_{\wedge\neg}} A$, then $\vdash_{INT} A$.

The proof is carried out by induction on the structure of $L_{\wedge\neg}$-formula A.

Now there are three options: *1)* A is a propositional variable p_i, *2)* A is ¬B, *3)* A is $\mathrm{B}_1 \wedge \mathrm{B}_2$.

Let's consider *1)*. Here the following statement requires provement — if $\vdash_{PC_{\wedge\neg}} p_i$, then $\vdash_{INT} p_i$. But none of propositional variables can be the theorem of *PC* calculi. Therefore, the theorem would be true here, taking into consideration the characteristics of the classical implication.

Let's consider *2)*. Here the following statement requires provement — if $\vdash_{PC_{\wedge\neg}} \neg B$, then $\vdash_{INT} \neg B$. But after Glivenko's result [9] that $\vdash_{PC} \neg A$ if and only if $\vdash_{INT} \neg A$ the previous statement would be true.

Let's consider *3)*. Here the following statement requires provement — if $\vdash_{PC_{\wedge\neg}} \mathrm{B}_1 \wedge \mathrm{B}_2$, then $\vdash_{INT} \mathrm{B}_1 \wedge \mathrm{B}_2$. Taking into consideration the characteristics of $PC_{\wedge\neg}$ it is true that *a)* if $\vdash_{PC_{\wedge\neg}} \mathrm{B}_1 \wedge \mathrm{B}_2$, then $\vdash_{PC_{\wedge\neg}} \mathrm{B}_1$ and $\vdash_{PC_{\wedge\neg}} \mathrm{B}_2$.

Using the inductive assumption, we have: *b)* if $\vdash_{PC_{\wedge\neg}} B_1$, then $\vdash_{INT} B_1$ and $\vdash_{PC_{\wedge\neg}} B_2$, then $\vdash_{INT} B_2$.

In *Int* the following formula is provable: *c)* $\mathrm{B}_1 \supset (\mathrm{B}_2 \supset (\mathrm{B}_1 \wedge \mathrm{B}_2))$. From *a)*, *b)* and *c)* by the definition of Int proof, we obtain that $\vdash_{INT} \mathrm{B}_1 \wedge \mathrm{B}_2$.

Thus, the theorem T2 is proved. □

Now let's introduce the result of Łukasiewicz [19] on embedment of PC into *Int*, when the binary identical relations of classical logic were used for connectives $\supset$ and $\vee$. This was initiated for representing the PC formulas, stated in the $L_{\wedge\vee\supset\neg}$ language, into the *Int* formulas. There has been constructed the following embedding operation (here and below we will use the original symbols for embedding operations):

- $p^* = p$
- $(A \wedge B)^* = A^* \wedge B^*$
- $(\neg A)^* = \neg(A^*)$
- $(A \vee B)^* = \neg((\neg A^*) \wedge (\neg B^*))$
- $(A \supset B)^* = \neg(A^* \wedge (\neg B^*))$.

Theorem 3 (T3). *$\vdash_{PC} A$ if and only if $\vdash_{INT} A^*$.*

According to Epstein (ref. [6, p. 213]), neither the embedment in the sense of T2, nor the embedment in the sense of T3 preserves the relations of consequences. That happens because these theorems are false in the wording of $G \vdash_{PC} A$ if and only if $G \vdash_{INT} A$. For example, if we have $\neg\neg p \vdash_{PC} p$, then we also have $\neg\neg p \vdash_{INT} p$, and hence we obtain by the theorem of deduction the false statement $\vdash_{INT} \neg\neg p \supset p$.

Proposed by Gentzen in 1936 [8], his embedment preserves the relation of consequences. This statement consists of the language $L(\neg, \supset, \wedge, \vee)$ and the following embedding operation:

- $p^\circ = \neg\neg p$
- $(A \wedge B)^\circ = A^\circ \wedge B^\circ$
- $(\neg A)^\circ = \neg(A^\circ)$
- $(A \vee B)^\circ = \neg((\neg A^\circ) \wedge (\neg B^\circ))$
- $(A \supset B)^\circ = A^\circ \supset B^\circ$,

THEOREM 4 (T4). *$G \vdash_{PC} A$ if and only if $G^\circ \vdash_{INT} A^\circ$.*

Here Gentzen managed to preserve the relation of consequences while embedment process exactly because of non-standard mapping of propositional variable (through double negation).

It also makes sense to specify here the aforementioned significant result of V.M. Popov [23] in order to close the review of the history of embedment of classical propositional logic into intuitionistic.

Classical propositional logic is axiomatized by calculus $Cl_{\supset f}$. Its axioms are those and only those $L_{\supset f}$-formulas, each of which is given as $A \supset (B \supset A)$ or $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$ or $((A \supset f) \supset f) \supset A$. The inference rule here: $A, A \supset B/B$.

Implicative fragment of intuitionistic propositional logic is axiomatized by calculus $Int_{\supset}$. Its axioms are those and only those $L_{\supset}$-formulas, each of which is given as $A \supset (B \supset A)$ or $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$. The inference rule here: $A, A \supset B/B$.

The following operations are offered here: *Sd* (first introduced by V.M. Popov in [24]), and *T*. *Sd* is meant to be a mapping of the set of all $L_{\supset f}$-formulas into the set of all $L_{\supset}$-formulas, and T is a mapping of the set of all $L_{\supset}$-formulas into the set of all $L_{\supset}$-formulas.

- $Sd(f) = p_1$,
- $Sd(p_i) = p_{i+1}$ (*where* $i \in \{1, 2, 3, \dots\}$),
- $Sd(A \supset B) = Sd(A) \supset Sd(B)$.
- $\mathrm{T}(p_1) = p_1$,
- $T(p_i) = (p_i \supset p_1) \supset p_1$ (*where* $i \in \{1, 2, 3, \dots\}$),
- $T(A \supset B) = T(A) \supset T(B)$.

Further the theorem provement takes place (T5 in our notation).

THEOREM 5 (T5). $\vdash_{Sd\supset f}$ A *if and only if* $\vdash_{Int\supset}$ T(Sd(A)).

V.A. Bocharov, M. Zaharyashev, V.I. Markin, A.V. Chagrov, L.L. Esakia should be also mentioned as the contemporary Russian scientists who gave their tribute to the study of logical systems through embedding operations. V.A. Bocharov in [1] constructs embedment of Boolean algebra into syllogistics, among other works on this subject. V.I. Markin in his book [20, pp. 35–43] embedded the systems of clear positive Aristotelian syllogistics into the predicate calculus. A translation from the calculus *RM* to the positive fragment of *RM* is constructed in [14]. Work of M. Zaharyashev and A.V. Chagrov [5] is dedicated to embedment of intuitionistic logic and its extensions to different normal modal logics. L.L. Esakia in [7] considers new aspects of Gödel's embedment of intuitionistic logic into modal logic *S4*.

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Syntax and semantics of simple paracomplete logics[1]

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abstract. For an arbitrary fixed element β in $\{1, 2, 3, \ldots \omega\}$ both a sequent calculus and a natural deduction calculus which axiomatise simple paracomplete logic $\mathrm{I}_{2,\beta}$ are built. Additionally, a valuation semantic which is adequate to logic $\mathrm{I}_{2,\beta}$ is constructed. For an arbitrary fixed element γ in $\{1, 2, 3, \ldots\}$ a cortege semantic which is adequate to logic $\mathrm{I}_{2,\gamma}$ is described. A number of results obtainable with the axiomatisations and semantics in question are formulated.

Keywords: paracomplete logic, paraconsistent logic, cortege semantics, valuation semantics, sequent calculus, natural deduction calculus

We study logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\ldots \mathrm{I}_{2,\omega}$ presented in [8]. These logics are paracomplete counterparts of paraconsistent logics $\mathrm{I}_{1,1}$, $\mathrm{I}_{1,2}$, $\mathrm{I}_{1,3}$, $\ldots \mathrm{I}_{1,\omega}$ from [7]. In the paper, (a) simple paracomplete logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\ldots \mathrm{I}_{2,\omega}$ are defined (see [8]); these logics form (in the order indicated above) a strictly decreasing (in terms of the set-theoretic inclusion) sequence of logics, (b) for any j in $\{0, 1, 2, 3, \ldots \omega\}$ both a sequent calculus $\mathrm{GI}_{2,j}$ (see [10]) and a natural deduction calculus $\mathrm{NI}_{2,j}$ which axiomatise logic $\mathrm{I}_{2,j}$ are formulated, (c) for any j in $\{1, 2, 3, \ldots \omega\}$, we propose a valuation semantics for logic $\mathrm{I}_{2,j}$ (see [9]), (d) for any j in $\{1, 2, 3, \ldots\}$, we propose a cortege semantics for logic $\mathrm{I}_{2,j}$ (see [9]). Below there are some results obtained with the semantics and calculi in question.

The language L of each logic in the paper is a standard propositional language with the following alphabet: $\{\&, \vee,$

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$\supset, \neg, (,), p_1, p_2, p_3, \dots\}$. As it is expected, &, $\vee$, $\supset$ are binary logical connectives in L, $\neg$ is a unary logical connective in L, brackets (,) are technical symbols in L and $p_1, p_2, p_3, \dots$ are propositional variables in L. A definition of L-formula is as usual. Below, we say 'formula' instead of 'L-formula' only and adopt the convention on omitting brackets as in [4]. A formula is said to be quasi-elemental iff no logical connective in L other than $\neg$ occurs in it. A length of a formula A is, traditionally, said to be the number of all occurrences of the logical connectives in L in A. We denote the rule of modus ponens in L by MP and the rule of substitution of a formula into a formula instead of a propositional variable in L by Sub. A logic is said to be a non-empty set of formulas closed under MP and Sub. A theory for logic **L** is said to be a set of formulas including logic **L** and closed under MP. It is understood that the set of all formulas is both a logic and a theory for any logic. The set of all formulas is said to be a trivial theory. A complete theory for logic **L** is said to be a theory T for logic **L** such that, for some formula A, $A \in$ T or $\neg A \in$ T. A paracomplete theory for logic **L** is said to be a theory T for logic **L** such that T is not a complete theory and any complete theory for logic **L**, which includes T, is a trivial theory. A paracomplete logic is said to be a logic **L** such that there exists a paracomplete theory for logic **L**. Simple paracomplete logic is said to be a paracomplete logic **L** such that for any paracomplete theory T for logic **L** holds true: there exists a quasi-elemental formula A such that neither A, nor $\neg A$ belongs to T.

Let us agree that anywhere in the paper: α is an arbitrary element in $\{0, 1, 2, 3, \dots \omega\}$, β is an arbitrary element in $\{1, 2, 3, \dots \omega\}$, γ is an arbitrary element in $\{1, 2, 3, \dots\}$. We define calculus $\mathrm{HI}_{2,\alpha}$. This calculus is Hilbert-type calculi, the language of $\mathrm{HI}_{2,\alpha}$ is L. $\mathrm{HI}_{2,\alpha}$ has MP as the only rule of inference. The notion of a derivation in $\mathrm{HI}_{2,\alpha}$ (of a proof in $\mathrm{HI}_{2,\alpha}$, in particular) is defined as usual; and for $\mathrm{HI}_{2,\alpha}$, both notion of a formula derivable from the set of formulas in this calculus and a notion of a formula provable in this calculus are defined as usual. Now we only need to define the set of axioms of $\mathrm{HI}_{2,\alpha}$.

A formula belongs to the set of axioms of calculus $\mathrm{HI}_{2,\alpha}$ iff it is one of the following forms (hereafter, A, B, C denote formulas):

(I) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$, (II) $A \supset (A \vee B)$, (III) $B \supset (A \vee B)$, (IV) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$, (V) $(A\&B) \supset A$, (VI) $(A\&B) \supset B$, (VII) $(C \supset A) \supset ((C \supset B) \supset (C \supset (A\&B)))$, (VIII) $(A \supset (B \supset C)) \supset ((A\&B) \supset C)$, (IX) $((A\&B) \supset C) \supset (A \supset (B \supset C))$, (X) $((A \supset B) \supset A) \supset A$, (XI, α) $(E \supset \neg(B \supset B)) \supset \neg E$, where E is formula which is not a quasi-elemental formula of a length less than α, (XII) $\neg A \supset (A \supset B)$.

Let us agree that, for any j in $\{0, 1, 2, 3, \dots \omega\}$, $\mathrm{I}_{2,j}$ is the set of formulas provable in $\mathrm{HI}_{2,j}$.

The following theorems 1 and 2 are shown.

THEOREM 1. *Sets $I_{2,0}$, $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\dots I_{2,\omega}$ are logics, and, for any k and l in $\{0, 1, 2, 3, \dots \omega\}$, if $k < l$, then $I_{2,l} \subseteq I_{2,k}$.*

THEOREM 2. *Logic $I_{2,0}$ is the set of the classical tautologies in L.*

Let us establish connections between logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\dots \mathrm{I}_{2,\omega}$ and logic $\mathrm{I}_{2,0}$ (that is, the classical propositional logic in L).

Let φ be a mapping of the set of all formulas into itself satisfying the following conditions: (1) $\varphi(p)$ is not a quasi-elemental formula, for any propositional variable p in L, (2) for any propositional variable p in L, formulas $p \supset \varphi(p)$ and $\varphi(p) \supset p$ belong to logic $\mathrm{I}_{2,0}$, (3) $\varphi(B \circ C) = \varphi(B) \circ \varphi(C)$, for any formulas B, C and for any binary logical connective $\circ$ in L, (4) $\varphi(\neg B) = \neg\varphi(B)$, for any formula B.

Following these conditions, theorem 3 is shown.

THEOREM 3. *For any j in $\{1, 2, 3, \dots \omega\}$ and for any formula A: $A \in I_{2,0}$ iff $\varphi(A) \in I_{2,j}$.*

Let now ψ be such a mapping the set of all formulas into itself satisfying the following conditions: (1) $\psi(p) = p$, for any propositional variable p in L, (2) $\psi(B \circ C) = \psi(B) \circ \psi(C)$, for any formulas B, C and for any binary logical connective $\circ$ in L, (3) $\psi(\neg B) = \psi(B) \supset \neg(p_1 \supset p_1)$, for any formula B.

Following these conditions, theorem 4 is shown.

THEOREM 4. *For any j in $\{1, 2, 3, \dots \omega\}$ and for any formula A: $A \in I_{2,0}$ iff $\psi(A) \in I_{2,j}$.*

Let us now show a method to build up a sequent calculus $\mathrm{GI}_{2,\beta}$ which axiomatises logic $\mathrm{I}_{2,\beta}$. Calculus $\mathrm{GI}_{2,\beta}$ (see [10]) is a Gentzen-type sequent calculus. Sequents are of the form $\Gamma \to \Delta$ (hereafter, Γ, Δ, Σ and Θ denote finite sequences of formulas). The set of basic sequents of $\mathrm{GI}_{2,\beta}$ is the set of all sequents of the form $A \to A$. The only rules of $\mathrm{GI}_{2,\beta}$ are the rules R1-R15, R16(β), R17 listed below.

$$\frac{\Gamma, A, B, \Delta \to \Theta}{\Gamma, B, A, \Delta \to \Theta}\ \mathrm{R1}, \quad \frac{\Gamma \to \Delta, A, B, \Theta}{\Gamma \to \Delta, B, A, \Theta}\ \mathrm{R2}, \quad \frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}\ \mathrm{R3},$$

$$\frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}\ \mathrm{R4}, \quad \frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}\ \mathrm{R5}, \quad \frac{\Gamma \to \Theta}{\Gamma \to \Theta, A}\ \mathrm{R6},$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Sigma \to \Theta}{A \supset B, \Gamma, \Sigma \to \Delta, \Theta}\ \mathrm{R7}, \quad \frac{A, \Gamma \to \Theta, B}{\Gamma \to \Theta, A \supset B}\ \mathrm{R8},$$

$$\frac{A, \Gamma \to \Theta}{A \& B, \Gamma \to \Theta}\ \mathrm{R9}, \quad \frac{A, \Gamma \to \Theta}{B \& A, \Gamma \to \Theta}\ \mathrm{R10}, \quad \frac{\Gamma \to \Theta, A \qquad \Gamma \to \Theta, B}{\Gamma \to \Theta, A \& B}\ \mathrm{R11},$$

$$\frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B}\ \mathrm{R12}, \quad \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, B \vee A}\ \mathrm{R13}, \quad \frac{A, \Gamma \to \Theta \qquad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}\ \mathrm{R14},$$

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ \mathrm{R15},$$

$$\frac{E, \Gamma \to \Theta}{\Gamma \to \Theta, \neg E}\ \mathrm{R16}(\beta),$$ where E is a formula which is not a quasi-elemental formula of a length less than β,

$$\frac{\Gamma \to \Delta, A \qquad A, \Sigma \to \Theta}{\Gamma, \Sigma \to \Delta, \Theta}\ \mathrm{R17\ (cut\ rule)}$$

A derivation in calculus $\mathrm{GI}_{2,\beta}$ is defined in a standard sequent calculus fashion. The definition of a sequent provable in $\mathrm{GI}_{2,\beta}$ is as usual. The cut-elimination theorem is shown (by Gentzen's method presented in [3]) to be valid in $\mathrm{GI}_{2,\beta}$.

The following theorem 5 is shown.

THEOREM 5. *For any j in $\{1, 2, 3, \ldots \omega\}$ and for any formula A: $A \in I_{2,j}$ iff a sequent $\to A$ is provable in $GI_{2,j}$.*

Let us now show a method to build up a Fitch-style natural deduction calculus $\mathrm{NI}_{2,\beta}$ which axiomatises logic $\mathrm{I}_{2,\beta}$.

The set of $\mathrm{NI}_{2,\beta}$-rules is as follows, where $[A]C$ denotes a derivation of a formula C from a formula A.

$$\frac{C \& C_1}{C}\ \&_{el1} \qquad \frac{C \& C_1}{C_1}\ \&_{el2} \quad \frac{C, C_1}{C \& C_1}\ \&_{in}$$

$$\frac{C \vee C_1,\ [C]C_2\ [C_1]C_2}{C_2} \vee_{el} \qquad \frac{C}{C \vee C_1} \vee_{in1} \qquad \frac{C_1}{C \vee C_1} \vee_{in2}$$

$$\frac{C \supset C_1,\ C}{C_1} \supset_{el} \qquad \frac{[C]C_1}{C \supset C_1} \supset_{in} \qquad \frac{[A \supset B]A}{A} \supset_{p}$$

$$\frac{[E]\ \neg(C \supset C)}{\neg E} \neg_{in1(\beta)},$$ where E is a formula which is not a quasi-elemental formula of a length less than β.

$$\frac{\neg C_1,\ C_1}{C} \neg_{in2}$$

A derivation in $\mathrm{NI}_{2,\beta}$ is defined in a standard natural deduction calculus fashion.

The following theorem 6 is shown.

THEOREM 6. *For any j in $\{1, 2, 3, \ldots \omega\}$ and for any formula A : $A \in I_{2,j}$ iff A is provable in $NI_{2,j}$.*

The proof search procedures which were proposed to the classical and a variety of non-classical logics are applicable [1, 2].

Let us construct $\mathrm{I}_{2,\beta}$-valuation semantics for $\mathrm{I}_{2,\beta}$. By Q_β we denote the set of all quasi-elemental formulas of a length less or equal to β. By $\mathrm{I}_{2,\beta}$-valuation we mean any mapping v set Q_β into the set $\{0,\ 1\}$ such that, for any quasi-elemental formula e of a length less than β, if $v(e) = 1$, then $v(\neg e) = 0$. Let Form denote the set of all formulas and let $\mathrm{Val}_{2,\beta}$ denote the set of all $\mathrm{I}_{2,\beta}$-valuations. It can be shown there exists a unique mapping (denoted by $\xi_{2,\beta}$) satisfying the following six conditions: (1) $\xi_{2,\beta}$ is a mapping a Cartesian product Form $\times$ $\mathrm{Val}_{2,\beta}$ into the set $\{1,\ 0\}$, (2) for any quasi-elemental formula Y in Q_β and any $\mathrm{I}_{2,\beta}$-valuation v: $\xi_{2,\beta}(Y, v) = v(Y)$, (3) for any formulas A, B and any $\mathrm{I}_{2,\beta}$-valuation v: $\xi_{2,\beta}(A\&B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ and $\xi_{2,\beta}(B) = 1$, (4) for any formulas A, B and any $\mathrm{I}_{2,\beta}$-valuation v: $\xi_{2,\beta}(A \vee B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ or $\xi_{2,\beta}(B) = 1$, (5) for any formulas A, B and any $\mathrm{I}_{2,\beta}$-valuation v: $\xi_{2,\beta}(A \supset B, v) = 1$ iff $\xi_{2,\beta}(A) = 0$ or $\xi_{2,\beta}(B) = 1$, (6) for any formula A which is not a quasi-elemental formula of a length less than β, and for any $\mathrm{I}_{2,\beta}$-valuation v: $\xi_{2,\beta}(\neg A, v) = 1$ iff $\xi_{2,\beta}(A, v) = 0$. A formula A is said to be $\mathrm{I}_{2,\beta}$-valid iff for any $\mathrm{I}_{2,\beta}$-valuation v, $\xi_{2,\beta}(A, v) = 1$.

The following theorems 7 and 8 are shown.

THEOREM 7. *For any j in $\{1, 2, 3, \ldots \omega\}$, for any formula A, for any set Γ of formulas: formula A is derivable from Γ in $HI_{2,j}$ iff for* *any $I_{2,j}$-valuation v, if for any formula B in Γ, $\xi_{2,j}(B,v)=1$, then $\xi_{2,j}(A,v)=1$.*

THEOREM 8. *For any j in $\{1,2,3,\dots\omega\}$ and for any formula A, $A \in I_{2,j}$ iff formula A is $I_{2,j}$-valid.*

It should be noted that the proposed $\mathrm{I}_{2,\beta}$-valuation semantics is consistent to the requirements, which, in our point of view, N.A. Vasiliev considers to be necessary in [11]: (1) no proposition cannot be true and false at once, (2) in general case, a value of the proposition that is a negation of a proposition P, is not determined by the value of P.

Let us construct $\mathrm{I}_{2,\gamma}$-cortege semantics for $\mathrm{I}_{2,\gamma}$. By $\mathrm{I}_{2,\gamma}$-cortege we mean an ordered $\gamma+1$-tuplet of elements of the set $\{1, 0\}$ such that for any two neighboring members of this ordered $\gamma+1$-tuplet, at least one of them is 0. By a designated $\mathrm{I}_{2,\gamma}$-cortege we mean $\mathrm{I}_{2,\gamma}$-cortege, where the first member is 1. By $\mathrm{S}_{2,\gamma}$ we denote the set of all $\mathrm{I}_{2,\gamma}$-corteges and by $\mathrm{D}_{2,\gamma}$ we denote the set of all designated $\mathrm{I}_{2,\gamma}$-corteges. By a normal $\mathrm{I}_{2,\gamma}$-cortege we mean $\mathrm{I}_{2,\gamma}$-cortege such that any two neighboring members of this $\mathrm{I}_{2,\gamma}$-cortege are different. By a single $\mathrm{I}_{2,\gamma}$-cortege we mean a normal $\mathrm{I}_{2,\gamma}$-cortege such that the first member of it is 1. By a zero $\mathrm{I}_{2,\gamma}$-cortege we mean a normal $\mathrm{I}_{2,\gamma}$-cortege such that the first member of it is 0.

It is clear that there exists a unique single $\mathrm{I}_{2,\gamma}$-cortege (denoted by $\mathbf{1}_\gamma$) and there exists a unique zero $\mathrm{I}_{2,\gamma}$-cortege (denoted by $\mathbf{0}_\gamma$). It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\&_{2,\gamma}$) satisfying the following condition, for any X, Y in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege X is 1 and the first member of $\mathrm{I}_{2,\gamma}$-cortege Y is 1 then $X \&_{2,\gamma} Y$ is $\mathbf{1}_\gamma$; otherwise, $X \&_{2,\gamma} Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\vee_{2,\gamma}$) satisfying the following condition, for any X and Y in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege X is 1 or the first member of $\mathrm{I}_{2,\gamma}$-cortege Y is 1 then $X \vee_{2,\gamma} Y$ is $\mathbf{1}_\gamma$; otherwise, $X \vee_{2,\gamma} Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\supset_{2,\gamma}$) satisfying the following condition, for any X and Y in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege X is 0 or the first member of $\mathrm{I}_{2,\gamma}$-cortege Y is 1 then $X \supset_{2,\gamma} Y$ is $\mathbf{1}_\gamma$; otherwise, $X \supset_{2,\gamma} Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique unary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\neg_{2,\gamma}$) satisfying the following condition, for any $\mathrm{I}_{2,\gamma}$-cortege $< x_1, x_2, \ldots, x_\gamma, x_{\gamma+1} >$: if $x_{\gamma+1}$ is 1 then $\neg_{2,\gamma}(< x_1, x_2, \ldots, x_\gamma, x_{\gamma+1} >) =< x_2, \ldots, x_\gamma, x_{\gamma+1}, 0 >$ and if, if $x_{\gamma+1}$ is 0, then $\neg_{2,\gamma}(< x_1, x_2, \ldots, x_\gamma, x_{\gamma+1} >) = < x_2, \ldots, x_\gamma, x_{\gamma+1}, 1 >$.

It is clear that $< \mathrm{S}_{2,\gamma}, \mathrm{D}_{2,\gamma}, \&_{2,\gamma}, \vee_{2,\gamma}, \supset_{2,\gamma}, \neg_{2,\gamma} >$ is a logical matrix. This logical matrix (denoted by $\mathrm{M}_{2,\gamma}$) is said to be $\mathrm{I}_{2,\gamma}$-matrix. $\mathrm{M}_{2,\gamma}$-valuation is said to be a mapping the set of all propositional variables in L into $\mathrm{S}_{2,\gamma}$. The set of all $\mathrm{M}_{2,\gamma}$-valuations is denoted by $\mathrm{ValM}_{2,\gamma}$. It can be shown that there exists a unique mapping (denoted by $\xi\mathrm{M}_{2,\gamma}$) satisfying the following conditions: (1) $\xi\mathrm{M}_{2,\gamma}$ is a mapping a Cartesian product Form x $\mathrm{ValM}_{2,\gamma}$ into the set $\mathrm{S}_{2,\gamma}$, (2) for any propositional variable p in L and for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(p, w) = w(p)$, (3) for any formulas A, B and for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(A\&B, w) = \xi\mathrm{M}_{2,\gamma}(A, w)\&_{2,\gamma}\xi\mathrm{M}_{2,\gamma}(B, w)$, (4) for any formulas A, B and for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(A \vee B, w) = \xi\mathrm{M}_{2,\gamma}(A, w) \vee_{2,\gamma} \xi\mathrm{M}_{2,\gamma}(B, w)$, (5) for any formulas A, B and for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(A \supset B, w) = \xi\mathrm{M}_{2,\gamma}(A, w) \supset_{2,\gamma} \xi\mathrm{M}_{2,\gamma}(B, w)$, (6) for any formula A and for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(\neg A, w) = \neg_{2,\gamma}\xi\mathrm{M}_{2,\gamma}(A, w)$.

A formula A is said to be $\mathrm{M}_{2,\gamma}$-valid iff for any $\mathrm{M}_{2,\gamma}$-valuation w, $\xi\mathrm{M}_{2,\gamma}(A, w) \in \mathrm{D}_{2,\gamma}$.

The following theorems 9–11 are shown.

THEOREM 9. *For any j in $\{1, 2, 3, \ldots\}$, for any formula A and for any set Γ of formulas, formula A is derivable from Γ in $HI_{2,j}$ iff for any $M_{2,j}$-valuation w, if for any formula B from Γ, $\xi M_{2,j}(B, w) \in D_{1,j}$ then $\xi M_{2,j}(A, w) \in D_{2,j}$.*

THEOREM 10. *For any j in $\{1, 2, 3, \ldots\}$ and for any formula A, $A \in I_{2,j}$ iff A is $M_{2,j}$-valid.*

THEOREM 11. *For any j in $\{1, 2, 3, \ldots\}$ and for any formula A, A is $M_{2,j}$-valid iff for any $M_{2,j}$-valuation w, $\xi M_{1,j}(A, w) \in \boldsymbol{1}_j$.*

The following theorems 12–19 are shown with the help of the axiomatisations and semantics presented in the paper.

THEOREM 12. *Logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\ldots I_{2,\omega}$ are simple paracomplete logics.*

THEOREM 13. *For any j and k in $\{1, 2, 3, \ldots \omega\}$, if $j \neq k$ then $I_{2,j} \neq I_{2,k}$.*

THEOREM 14. *For any j in $\{1, 2, 3, \ldots \omega\}$, the positive fragment of logic $I_{2,j}$ is equal to the positive fragment of logic $I_{2,0}$.*

THEOREM 15. *For any j in $\{1, 2, 3, \ldots \omega\}$, logic $I_{2,j}$ is decidable.*

THEOREM 16. *For any j in $\{1, 2, 3, \ldots\}$, logic $I_{2,j}$ is finitely-valued.*

THEOREM 17. *Logic $I_{2,\omega}$ is not finitely-valued.*

THEOREM 18. *Logic $I_{2,\omega}$ is equal to the intersection of logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\ldots$*

THEOREM 19. *There is a continuum of logics which include $I_{2,\omega}$ and are included in $I_{2,1}$.*

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Cardinality of sets of closed functional classes in weak 3-valued logics

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abstract. This paper proves that sets of closed functional classes in 3-valued logics of Bochvar B_3 and Hallden H_3 contains a continuum of different closed classes. It is also proven that both of these logics contain a closed functional class which has no basis.

Keywords: Bochvar's logic, Hallden's logic, closed class, continuum, cardinality

The research on cardinality of closed sets of functions in different logics was started by E. Post. Thus, in [10] he proved that classical logic only contains an enumerable set of different closed functional classes. In 1959 Yu.I. Yanov and A.A. Muchnik [3] for the first time showed that for every $k \geq 3$ the k-valued Post's logic P_k contains a closed class which has no basis, and also contains a continual set of different closed functional classes. M.F. Ratsa in [4] and [5] showed that 3-valued logic of Heyting G_3 contains a continual set of different functional classes which have bases and a continual set of classes which have no bases. Consequently, cardinalities of sets of closed functional classes in different logics were researched in the fundamental monograph [9] by D. Lau which deals with functional algebras on finite sets.

A.S. Karpenko in [2] suggested a hypothesis that the set of closed classes in Bochvar's 3-valued logic B_3 has the power of continuum (truth-tables, defining basic functions of B_3, will be formulated below). As a justification of this hypothesis the author uses the condition (see [9, pp. 221–222]) for a class to contain just an enumerable set of closed functional classes. A.S. Karpenko found out that logic B_3 does not satisfy this criterion. Nevertheless, this condition is a sufficient but not necessary one. Despite this argument, we cannot conclude that the set of closed classes in B_3 is continual.

Thus, the question on cardinality of the set of closed classes in B_3 remained open until now. It was also unknown, whether there are functional classes in B_3 (id est logics weaker, than B_3 itself) which contain continual sets of closed classes. In this paper the author gives positive answers to these questions, and the answer to the latter may also be viewed as an answer to the former of them. In particular, as 3-valued Hallden's logic H_3 which was first studied in [8] contains (as shown below) a continual set of different closed classes, and as H_3 is precomplete in B_3 (this fact was proven by V.K. Finn in [6]), the set of closed classes in B_3 is continual.

Let us formulate a series of corresponding theorems and prove them. For this purpose we shall use the slightly modified strategy of Yu.I. Yanov and A.A. Muchnik.

Below we shall use basic functions of B_3, defined by the following truth-tables:

$\cap$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$\cup$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\sim$	x
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\vdash$	x
1	1
0	$\frac{1}{2}$
0	0

THEOREM 1. *The set of closed classes in B_3 contains a class, which has no basis.*

PROOF. Let us consider a sequence

$$S = f_0, f_1(x_1), f_2(x_1, x_2), ...$$

of functions $f_i(x_1, ..., x_i)$ of 3-valued Post logic P_3 for $i \in \{0, 1, 2, ...\}$, satisfying the following conditions:

$$f_0 \equiv 0;$$

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{if } x_1 = ... = x_i = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

First of all, it is necessary to demonstrate that all the functions $f_i(x_1, ..., x_i)$ are in B_3. For this purpose it is sufficient to observe that the constant 0 is in B_3 and may be expressed with, for example, a formula $\vdash x \cap \sim\vdash x$, and to express the rest of the functions $f_i(x_1, ..., x_i)$ for every $i > 0$ with formulas:

$$\sim\vdash x_1 \cap \sim\vdash\sim x_1 \cap ... \cap \sim\vdash x_i \cap \sim\vdash\sim x_i.$$

It is worth mentioning that the formula $\sim\vdash x \cap \sim\vdash\sim x$ represents the Rosser–Turquette operator (J-operator) for the value $\frac{1}{2}$ in B_3[1]. So, the formula, expressing functions $f_i(x_1, ..., x_i)$, may be simplified with the use of J-operator for the value $\frac{1}{2}$ just as follows:

$$J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_i).$$

Let $\mathfrak{M}(S)$ be a class generated by the set of functions

$$\{f_0, f_1(x_1), f_2(x_1, x_2), ...\} \subset B_3$$

by renaming variables without identifying them. This class is a closed one. Let us also assume that $\mathfrak{M}(S)$ has a basis. In this case, there is a function f' that is obtained from function $f_{n_0}(x_1, ..., x_{n_0})$ through renaming variables for which the number n_0 is minimal. Then we have two cases:

1. The basis contains at least one more function f'' corresponding to a function $f_{n_1}(x_1, ..., x_{n_1})$ with $n_1 > n_0$. As $f_{n_0}(x_1, ..., x_{n_0})$ may be obtained from $f_{n_1}(x_1, ..., x_{n_1})$ by identifying some of the variables $x_1, ..., x_{n_1}$, the function f' may be expressed through f'', and this contradicts to the definition of a basis.

2. The basis consists of a single function f'. In this case no other function f_n for $n > n_0$ can be expressed with f', as $f_{n_0}(..., f_{n_0}, ...) \equiv 0$, that leads to a contradiction again. □

Theorem 2. *There is a closed class with an enumerable basis in B_3.*

Proof. To prove the theorem we shall consider a sequence

$$S = f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...$$

[1]RosserTurquette operators $J_1(x) = \vdash x$, $J_{\frac{1}{2}}(x) = \sim\vdash x \cap \sim\vdash\sim x$ and $J_0(x) = \vdash\sim x$ for B_3 were for the first time constructed by V.K. Finn in [7].

of functions $f_i(x_1, ..., x_i)$ in 3-valued logic of Post P_3 for $i \in \{2, 3, ...\}$, which satisfy the following conditions:

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{for } x_1 = ... = x_{j-1} = x_{j+1} = ... = x_i = \frac{1}{2}, \\ & x_j = 1 \ (1 \leq j \leq i); \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that for every i these functions can be defined using the basic functions of B_3. With this purpose for every x_j $(1 \leq j \leq i)$ and every i we shall consider the following formulae of B_3:

$F_j = \vdash x_j \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1) \cap ... \cap (\sim\vdash x_{j-1} \cap \sim\vdash\sim x_{j-1}) \cap (\sim\vdash x_{j+1} \cap \sim\vdash\sim x_{j+1}) \cap ... \cap (\sim\vdash x_i \cap \sim\vdash\sim x_i)$.

Then let F be the internal disjunction of formulae F_j:

$$F = \bigcup_{1}^{i} F_j.$$

For every fixed i, formulae F define functions $f_i(x_1, ..., x_i)$ from B_3. Thus, for example, for $i = 2$, there are only two formulae F_j, id est: $F_1 = \vdash x_1 \cap (\sim\vdash x_2 \cap \sim\vdash\sim x_2)$ and $F_2 = \vdash x_2 \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1)$. Then $F = F_1 \cup F_2$ is expressed in the following manner:

$(\vdash x_1 \cap (\sim\vdash x_2 \cap \sim\vdash\sim x_2)) \cup (\vdash x_2 \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1))$.

It is easy to verify that the function $f_2(x_1, x_2) \in B_3$ corresponding to this formula returns the value 1 only on two tuples $\langle 1, \frac{1}{2} \rangle$ and $\langle \frac{1}{2}, 1 \rangle$ of truth-values of variables x_1 and x_2. On all other tuples of truth-values this function returns the value 0.

Thus, it is proven that for every i functions $f_i(x_1, ..., x_i)$ are in B_3.

Notation of formulae F_j and F may be simplified essentially if we use J-operators for the truth-values 1 and $\frac{1}{2}$:

$$F'_j = J_1(x_j) \cap J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_{j-1}) \cap J_{\frac{1}{2}}(x_{j+1}) \cap ... \cap J_{\frac{1}{2}}(x_i).$$

Formula F, in this case, should be rewritten as the internal disjunction of all of the F'_j:

$$F' = \bigcup_{1}^{i} F'_j.$$

Let $\mathfrak{M}(S)$ be a closed class generated by the system of functions $\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$. We shall prove that this system is a basis for $\mathfrak{M}(S)$. It is sufficient to show that none of the functions $f_m(x_1, ..., x_m)$ in this class can be expressed only with functions

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\},$$

id est there is no representation:

$$f_m(x_1, ..., x_m) = \mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...].$$

The formula $\mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...]$ may be rewritten as:

$$\mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...] =$$
$$= f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]).$$

Using the first equation, we have:

$$f_m(x_1, ..., x_m) =$$
$$= f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]).$$

Let us observe three possible cases:

1. At least two of the formulae among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...],$$

where $r \geq 2$, do not coincide with symbols of variables. In this case, for every m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values of variables $x_1, ..., x_m$, there are values 1 or 0 on corresponding argument places of the function

$$f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]),$$

and this function, according to its definition, will be equivalent to 0. That is a contradiction to the hypothesis that the function $f_m(x_1, ..., x_m)$ may be expressed only with functions from

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\},$$

as no function in the set

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$$

is equivalent to 0.

2. Only one formula $\mathfrak{B}_s$ among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]$$

does not coincide with a symbol of variable. In this case, functions corresponding to the rest of the formulae in this list are equivalent to variables, and, as $r \geq 2$, there is at least one formula $\mathfrak{B}_p \equiv x_q$. Let us consider an m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ such that $\alpha_1 = ... = \alpha_{q-1} = \alpha_{q+1} = ... = \alpha_m = \frac{1}{2}$, and $\alpha_q = 1$. On this ordered set of truth-values the function corresponding to the formula $\mathfrak{B}_s$ returns the value 1 or 0. Then on the m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ in the function

$$f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...])$$

there are at least two argument places having truth-values which do not coincide with $\frac{1}{2}$. Therefore, the right part of the equation is equal to 0, and its left part must, according to definition of the function $f_m(x_1, ..., x_m)$, be equal to 1 which is impossible.

3. All of the formulae among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]$$

are equivalent to symbols of variables. Then $r > m$, and there are at least two entries of some variable x_p in the formula expressing the function $f_m(x_1, ..., x_m)$. Considering the ordered m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ such that $\alpha_1 = ... = \alpha_{p-1} = \alpha_{p+1} = ... = \alpha_m = \frac{1}{2}$ and $\alpha_p = 1$, we find out again that the right part of the corresponding equation is equal to 0, and its left part is equal to 1 which is impossible.

All three cases lead to contradiction. Therefore, none of the functions $f_m(x_1, ..., x_m)$ where $m \geq 2$ can be represented as a formula using only functions from

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\}.$$

□

This theorem allows to prove one of the main results of this paper, that the set of closed classes in B_3 has the cardinality of continuum. The method for proving this result is the same with the strategy used by Yu.I. Yanov and A.A Muchnik, to prove continuality of the set of closed classes of functions in k-valued logics of Post P_k, for all $k \geq 3$.

THEOREM 3. *The class of functions of B_3 contains a continuum of different closed sets.*

PROOF. The upper bound for cardinality of the set of closed classes in B_3 coincides with cardinality of the set of all subsets of functions in B_3. As the set of functions in B_3 is enumerably infinite, the set of all subsets of this set has the cardinality of continuum.

To obtain the lower bound for cardinality of the set of closed classes in B_3 it is enough to consider the closed class $\mathfrak{M}(S)$ constructed in the previous theorem. This class has a basis

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}.$$

For every sequence $S' = s_1, s_2, ...$ of natural numbers, where $2 \leq s_1 < s_2 < ...$, let us consider a closed class $\mathfrak{M}(S')$ which has a following set of functions as its basis:

$$\{f_{s_1}(x_1, ..., x_{s_1}), f_{s_2}(x_1, ..., x_{s_2}), ...\}.$$

It is obvious that

$$\mathfrak{M}(s_1, s_2, ...) \neq \mathfrak{M}(s'_1, s'_2, ...),$$

if $\{s_1, s_2, ...\} \neq \{s'_1, s'_2, ...\}$.

Consequently, the set of closed classes $\{\mathfrak{M}(S')\}$ in the set of closed classes of B_3 is continual. □

A question arises, whether existence of a set

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$$

of functions defined in the previous manner in a functional class of some 3-valued logic is necessary for this logic to contain a continuum of different closed classes. In general, the answer to this question is 'wrong', as the above-formulated definition of the sequence S of functions $f_i(x_1, ..., x_i)$ $(i \geq 2)$ is, in a certain sense, too strong, because it requires possibility to use at least two J-operators, as we can do in B_3. But prerequisites of this definition may be weakened essentially, so that we shall be able to prove one of the key theorems of this paper about continuality of the set of closed classes in 3-valued logic of Hallden H_3.

The basic connectives of logic H_3 are those in the set $\{\sim, J_{\frac{1}{2}}, \cap, \cup\}$ (for example, see [1, p. 57]).

THEOREM 4. *The set of closed functional classes in H_3 contains a class, which has no basis.*

PROOF. The sequence of functions

$$S = f_0, f_1(x_1), f_2(x_1, x_2), ...$$

is defined, using Rosser–Turquette operator $J_{\frac{1}{2}}(x)$, just as it was done in Theorem 1. The rest of the proof is completely analogous to the proof of Theorem 1. $\square$

THEOREM 5. *The class of functions in H_3 contains a closed class, which has an enumerable basis.*

PROOF. To prove the theorem, consider a sequence

$$S = f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...$$

of functions $f_i(x_1, ..., x_i)$ of 3-valued logic of Post P_3, for $i \in \{2, 3, ...\}$, satisfying the following definition:

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{if } x_1 = ... = x_{j-1} = x_{j+1} = ... = x_i = \frac{1}{2}, \\ & x_j \in \{1, 0\} \ (1 \leq j \leq i); \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that such functions may be defined using the basic functions of H_3, for each i. With this purpose we need to consider, for every x_j $(1 \leq j \leq i)$ and every i, the following formulae of H_3:

$F_j =\sim J_{\frac{1}{2}}(x_j) \cap J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_{j-1}) \cap J_{\frac{1}{2}}(x_{j+1}) \cap ... \cap J_{\frac{1}{2}}(x_i)$.
Then, for every i, let F be the internal disjunction of all of the formulae F_j:

$$F = \bigcup_{1}^{i} F_j.$$

The rest of the proof is analogous to the corresponding proof for B_3. □

THEOREM 6. *There is a continuum of different closed functional classes among functions of H_3.*

PROOF. The proof of this theorem is identical with the corresponding proof for B_3. □

It is worth mentioning that proofs of theorems 4–6 do not depend on proofs of theorems 1–3, and, as H_3 is precomplete in B_3, the former of them may be viewed as independent proofs of corresponding facts for B_3.

After proving these theorems one can suppose that enjoying the property of having a continuum of different closed functional classes for various multi-valued logics is rather a normal phenomenon, than a strange deviation, even for very weak multi-valued functional systems like H_3. If this hypothesis is true, it may be viewed as a new philosophical argument enforcing the thesis about qualitative difference between multi-valued logics and classical bivalent logic.

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Natural three-valued logics and classical logic

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abstract. In this paper implicative fragments of natural three-valued logic are investigated. It is proved that some fragments are equivalent by set of tautologies to implicative fragment of classical logic. It is also shown that some natural three-valued logics verify all tautologies of classical propositional logic.

Keywords: three-valued logis, natural implication, classical logic, set of tautologies

1 Introduction

In paper [3] we investigated functional properties of three-valued logics. We define some conditions for 'good' implication and introduce the idea of *natural* implication. So, as the result we have class of 30 implications[1] with strictly specified *natural* properties. Extensions of regular Kleene's logics by natural implications were regarded.

According to our definition, natural three-valued logic is a logic which includes natural implication as a connective.

On examination of 30 implicative extensions of weak Kleene's logic we received 7 *basic* logics[2]: Łukasiewicz's logic $\mathbf{Ł_3}$, paraconsistent logic **PCont**, three-valued Bochvar's logic $\mathbf{B_3}$, logic **Z**, $\mathbf{T^3}$, $\mathbf{T^2}$ and $\mathbf{T^1}$. These logics form a lattice w.r.t. relation of functional inclusion one logic to another.

Thus all these different three-valued systems, which appeared historicaly on different motivations, are presented in the same language

[1] Truth-tables for *natural* implications are given in appendix.

[2] In [4] the functional eqiuvality of some implicative extensions of weak Kleene's logic was proved.

with the following connectives: $\sim$, $\cup$ and $\rightarrow$, where $\sim$, $\cup$ — connectives of weak Kleene's logic and $\rightarrow$ — *natural* implication. It will allow us to compare these logics by set of tautologies. This is the next point of our research. And in this paper we focus on implicative fragments of natural three-valued logics.

2 Basic definitions

For the sake of clarity let us formulate some basic definitions.

Definition 1. The language $L_{\rightarrow}$ is a propositional language with the following alphabet:

(1) $p, q, r, \ldots$ — propositional variables;

(2) $\rightarrow$ — binary logical connective;

(3) $($, $)$ — technical symbols.

Definition 2. A definition of $L_{\rightarrow}$-formula is as usual:

(1) if A is propositional variable, then A is $L_{\rightarrow}$-formula;

(2) if A and B are $L_{\rightarrow}$-formulas, then $A \rightarrow B$ is $L_{\rightarrow}$-formula;

(3) nothing else is $L_{\rightarrow}$-formula.

Definition 3. A logical matrix is a structure $\mathfrak{M} =< V, F, D >$, where V is the set of truth-values, F is a set of functions on V called *basic functions*, and D is a set of designated values, D is a subset of V.

In this paper we will concider the logical matrices, where $V = \{1, 1/2, 0\}$ (let denote this set as V_3), F consists of one function[3] — *natural* implication and $D = \{1\}$ or $D = \{1, 1/2\}$.

Let's recall definition of *natural* implication:

Definition 4. Implication is called *natural* if it is satisfied the following criteria:

(1) **C**-extending, i.e. restrictions to the subset $\{0, 1\}$ of V_3 coincide with the classical implication.

[3]When we consider the implicative fragments of natural three-valued logics.

(2) If $p \to q \in D$ and $p \in D$, then $q \in D$, i.e. matrices for implication need to be normal in the sense of Łukasiewicz-Tarski (they verify the modus ponens) [2, p. 134].

(3) Let $p \leq q$, then $p \to q \in D$.

(4) $p \to q \in V_3$, in other cases.

According to the definition of *natural* implication, there are 6 implications with $D = \{1\}$ and 24 implications with $D = \{1, 1/2\}$ (appropriate truth-tables are given in appendix).

DEFINITION 5. A valuation v of an arbitrary $L_{\to}$-formula A in $\mathfrak{M}$ (symbolically — $|A|_v^{\mathfrak{M}}$) is defined as usual: $|p|_v^{\mathfrak{M}} \in V_3$, if p is a propositional variable; if A and B are $L_{\to}$-formulas, and $\to$ is basic function in $\mathfrak{M}$, then $|A \to B|_v^{\mathfrak{M}} = |A|_v^{\mathfrak{M}} \to |B|_v^{\mathfrak{M}}$.[4]

DEFINITION 6. An arbitrary $L_{\to}$-formula A is a *tautologie* in $\mathfrak{M}$ iff $|A|_v^{\mathfrak{M}} \in D$ for all valuation v in $\mathfrak{M}$.

3 Implicative fragments of natural three-valued logics

Let consider the following matrices which corespond to the implicative fragments of natural three-valued logics:

$$\mathfrak{M}^i_{\to} =< \{1, 1/2, 0\}, \to_i, \{1\} >, \; i \in \{1, 2, 3, 4, 5, 6\},$$

$$\mathfrak{M}^i_{\to} =< \{1, 1/2, 0\}, \to_i, \{1, 1/2\} >, \; i \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\},$$

$$\mathfrak{M}^{1'}_{\to} =< \{1, 1/2, 0\}, \to_1, \{1, 1/2\} >,$$

$$\mathfrak{M}^{4'}_{\to} =< \{1, 1/2, 0\}, \to_4, \{1, 1/2\} >,$$

where matrix operation $\to$ is defined by appropriate truth-tables of *natural* implications.

The following tautologies express the fundamental properties of implication:

[4] For the clarity we use the same symbols both for language functor (propositional connective) and corresponding matrix function.

$K:\ p \to (q \to p)$

$S:\ (p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$S':\ ((p \to q) \to r) \to ((p \to r) \to (q \to r))$

$P:\ ((p \to q) \to p) \to p$

$W:\ (p \to (p \to q)) \to (p \to q)$

$C:\ (p \to (q \to r)) \to (q \to (p \to r))$

$B:\ (q \to r) \to ((p \to q) \to (p \to r))$

And all implicative fragments of natural three-valed logics can be divided into 10 classes according to the fact that implicative formulas are tautologies in corresponding matrices:

	K	S	S'	P	W	C	B
$\mathfrak{M}^{1'}_{\to}, \mathfrak{M}^{i}_{\to}$ $(i \in \{2, 5, 7, 8, 9, 10, 11, 12, 13$ $17, 18, 19, 20, 21, 22, 23, 24\})$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\mathfrak{M}^{1}_{\to}$	$+$	$+$	$+$	$-$	$+$	$+$	$+$
$\mathfrak{M}^{25}_{\to}$	$-$	$+$	$+$	$-$	$+$	$+$	$+$
$\mathfrak{M}^{3}_{\to}$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
$\mathfrak{M}^{4}_{\to}$, $\mathfrak{M}^{4'}_{\to}$	$-$	$+$	$+$	$-$	$+$	$-$	$+$
$\mathfrak{M}^{27}_{\to}$	$-$	$-$	$+$	$+$	$+$	$-$	$-$
$\mathfrak{M}^{26}_{\to}$	$-$	$+$	$-$	$-$	$+$	$-$	$-$
$\mathfrak{M}^{i}_{\to}$ $(i \in \{15, 28\})$	$-$	$-$	$-$	$+$	$+$	$-$	$-$
$\mathfrak{M}^{6}_{\to}$	$-$	$-$	$+$	$-$	$-$	$-$	$-$
$\mathfrak{M}^{i}_{\to}$ $(i \in \{14, 16\})$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

So, let us consider the class matrices (corresponding to the first line of table above), in which all given implicative formulas are tautologies. This class consists of 18 matrices: 2 with $D = \{1\}$ and 16 with $D = \{1, {}^{1}\!/_{2}\}$. We can prove that all these matrices have the same class of tautologies.

The reasoning is as follows. For example, consider the matrices:

$\mathfrak{M}^7_{\rightarrow} =< \{1, 1/2, 0\}, \rightarrow_7, \{1, 1/2\} >$

$\mathfrak{M}^{13}_{\rightarrow} =< \{1, 1/2, 0\}, \rightarrow_{13}, \{1, 1/2\} >$

To show that these matrices have the same class of tautologies is sufficent to prove the following theorem:

THEOREM 1. *For any $L_{\rightarrow}$-formula A, for any valuation v*[5]*:*

$$|A|_v^{\mathfrak{M}^7_{\rightarrow}} = 0 \text{ iff } |A|_v^{\mathfrak{M}^{13}_{\rightarrow}} = 0.$$

PROOF may be given by induction on the structure of formula A.

Base case. Let A is a propositional variable, then it is obvious that theorem is true for this case.

Induction step. Let us assume that theorem is true for the formulas, that contain less than n occurrence of propsitional connectives (*induction hypothesis*). Then it is sufficent to prove, that theorem is true for $L_{\rightarrow}$-formula A that contains precisely n occurrence of propsitional connectives and graphically identical with formula $(B \rightarrow C)$, i.e. $A \mathrel{=\!\!=} (B \rightarrow C)$.

Then, the proof of the theorem reduces to the proof of the following two propositions:

PROPOSITION 1. $\forall v \forall A$: *if* $|A|_v^{\mathfrak{M}^7_{\rightarrow}} = 0$*, then* $|A|_v^{\mathfrak{M}^{13}_{\rightarrow}} = 0$.

PROPOSITION 2. $\forall v \forall A$: *if* $|A|_v^{\mathfrak{M}^{13}_{\rightarrow}} = 0$*, then* $|A|_v^{\mathfrak{M}^7_{\rightarrow}} = 0$.

Let us present the proof of the Proposition 1.

PROOF.

1.	Let proposition 1 does not hold	– assumption
2.	$\exists v \exists A : |A|_v^{\mathfrak{M}^7_{\rightarrow}} = 0$ and $|A|_v^{\mathfrak{M}^{13}_{\rightarrow}} \neq 0$	– from 1
3.	$|B \rightarrow C|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$ and $|B \rightarrow C|_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} \neq 0$	– from 2, elimination of quantifiers
4.	$|B \rightarrow C|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$	– from 3
5.	$|B|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} \rightarrow_7 |C|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$	– from 4, def. 5
6.	$|B|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} \in \{1, 1/2\}$ and $|C|_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$	– from 5, def. of $\rightarrow_7$

[5]As set V_3 in $\mathfrak{M}^7_{\rightarrow}$ and in $\mathfrak{M}^{13}_{\rightarrow}$ is the same, then it is true that any valuation in $\mathfrak{M}^7_{\rightarrow}$ is valuation in $\mathfrak{M}^{13}_{\rightarrow}$ and vice versa.

7.	$\lvert C\rvert_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$	– from 6
8.	$\lvert C\rvert_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} = 0$	– from 7 by induction hypothesis
9.	$\lvert B \rightarrow C\rvert_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} \neq 0$	– from 3
10.	$\lvert B\rvert_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} \rightarrow_{13} \lvert C\rvert_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} \neq 0$	– from 9
11.	$\lvert B\rvert_{v^*}^{\mathfrak{M}^{13}_{\rightarrow}} = 0$	– from 10 and 8, def. of $\rightarrow_{13}$
12.	$\lvert B\rvert_{v^*}^{\mathfrak{M}^7_{\rightarrow}} = 0$	– from 11 by induction hypothesis
13.	$\lvert B\rvert_{v^*}^{\mathfrak{M}^7_{\rightarrow}} \in \{1, 1/2\}$	– from 6
14.	$\lvert B\rvert_{v^*}^{\mathfrak{M}^7_{\rightarrow}} \neq 0$	– from 13
15.	Assumption 1. is incorrect	– from 12 and 14

Proposition 1 is proved. □

The proof of Proposition 2 is analogous to that of Proposition 1. Thus theorem is proved. □

By using similar methods of reasoning, it is not difficult to prove that all 18 matrices (matrices of the first group) have the same set of tautologies.

Let us investigate these 18 matrices in detail. It is well known that the implicative fragment of classical logic can be characterized deductively by the axioms K, S and P and the inference rule *modus ponens*. From this point of view each of 18 implcative fragments discussed above are the classical ones.

Let us consider natural three-valued logics, which implicative fragments are equivalent to the implicative fragment of classical logic. Corresponding logical matrices are the following:

$$\mathfrak{M}^i =< \{1, 1/2, 0\}, \sim, \cup, \rightarrow_i, \{1\} >,\ i \in \{1, 5\},$$

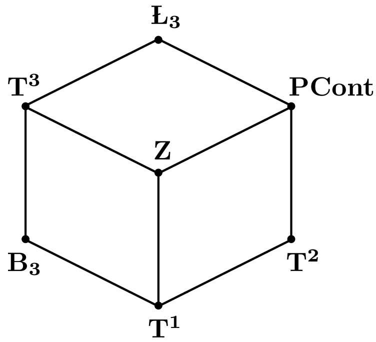
$$\mathfrak{M}^i =< \{1, 1/2, 0\}, \sim, \cup, \rightarrow_i, \{1, 1/2\} >,$$
$$i \in \{2, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 20, 21, 22, 23, 24\},$$

where $\sim$, $\cup$ are defined like in weak Kleene's logic, appropriate truth-tables for $\rightarrow_i$ are given in appendix.

From functional point of view, these 18 systems correspond to 7 *basic* logic:

$\mathbf{Ł_3}$	**PCont**	$\mathbf{B_3}$	**Z**	$\mathbf{T^1}$	$\mathbf{T^2}$	$\mathbf{T^3}$
$\mathfrak{M}^i$ $(i \in \{1,2,8,9,$ $10,11,12\}$	$\mathfrak{M}^i$ $(i \in \{18,19,$ $20,21,22\}$	$\mathfrak{M}^i$ $(i \in \{5,7\})$	$\mathfrak{M}^{17}$	$\mathfrak{M}^{23}$	$\mathfrak{M}^{24}$	$\mathfrak{M}^{13}$

7 *basic* logics form the following lattice w.r.t. relation of functional inclusion one logic to another:



Let us show that a constant $\perp$, which interpreted as falsehood, is defined by the basic functions of the 10 matrices $\mathfrak{M}^i$, $(i \in \{1,2,5,7,8,9,10,11,12,13\})$:

$$\perp = \sim (p \to_i p),\ i \in \{1,2,5,7,8,9,10,11,12,13\}.$$

But as follows from Wajsberg's work [5, § 5] the addition of $\perp \to p$ to the axiomatization of implicative fragment of classical logic gives the full classical propositional logic. Thus 10 natural three-valued logics, considered above, verify all tautologies of classical propositional logic.

Remark. In [1, p. 54] by using a computer program it was calculated that there are 18 C-extending isomorphs of classical logic, which verify *modus ponens*. So, it was found that in matirces corresponding to these isomorphs one of the basic functions — implicative function, is defined precisely by the same truth-tables of *natural* implications, as in 18 natural three-valued logics mentioned above.

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Appendix. Truth-tables for *natural* implications

Let us give truth-tables for *natural* implications according to the definition 4.

There are 6 implications with $D = \{1\}$ and 24 implications with $D = \{1, 1/2\}$. Note, that 2 paires of implications ($\to_1$ and $\to_4$ in the proposed list below) are the same with $D = \{1\}$ and $D = \{1, 1/2\}$.

$D = \{1\}$

$\to_1$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1	1

$\to_2$	1	1/2	0
1	1	1/2	0
1/2	1	1	1
0	1	1	1

$\to_3$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

$\to_4$	1	1/2	0
1	1	0	0
1/2	1	1	0
0	1	1	1

$\to_5$	1	1/2	0
1	1	0	0
1/2	1	1	1
0	1	1	1

$\to_6$	1	1/2	0
1	1	0	0
1/2	1	1	1/2
0	1	1	1

$D = \{1, 1/2\}$

$\to_7$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

$\to_8$	1	1/2	0
1	1	1	0
1/2	1/2	1	0
0	1	1	1

$\to_9$	1	1/2	0
1	1	1	0
1/2	1/2	1	0
0	1	1/2	1

$\to_{10}$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1/2	1

$\to_{11}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1	0
0	1	1	1

$\to_{12}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1	0
0	1	1/2	1

$\to_{13}$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1/2	1

$\to_{14}$	1	1/2	0
1	1	0	0
1/2	1/2	1	0
0	1	1	1

$\to_{15}$	1	1/2	0
1	1	0	0
1/2	1	1	0
0	1	1/2	1

$\to_{16}$	1	1/2	0
1	1	0	0
1/2	1/2	1	0
0	1	1/2	1

$\to_{17}$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	1	1	1

$\to_{18}$	1	1/2	0
1	1	1	0
1/2	1/2	1/2	0
0	1	1	1

$\to_{19}$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	1	1/2	1

$\to_{20}$	1	1/2	0
1	1	1	0
1/2	1/2	1/2	0
0	1	1/2	1

$\to_{21}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1	1

$\to_{22}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	1	1	1

$\to_{23}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1/2	1

$\to_{24}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	1	1/2	1

$\to_{25}$	1	1/2	0
1	1	0	0
1/2	1	1/2	0
0	1	1	1

$\to_{26}$	1	1/2	0
1	1	0	0
1/2	1/2	1/2	0
0	1	1	1

$\to_{27}$	1	1/2	0
1	1	0	0
1/2	1	1/2	0
0	1	1/2	1

$\to_{28}$	1	1/2	0
1	1	0	0
1/2	1/2	1/2	0
0	1	1/2	1

Dialogue games for Dishkant's quantum modal logic

Vladimir L. Vasyukov[1]

abstract. Recently some elaborations were made concerning the game theoretic semantic of $Ł_{\aleph_0}$ and its extension. In the paper this kind of semantics is developed for Dishkant's quantum modal logic ŁQ which is also, in fact, the specific extension of $Ł_{\aleph_0}$. As a starting point some game theoretic interpretation for the SŁ system (extending both Łukasiewicz logic $Ł_{\aleph_0}$ and modal logic $S5$) was exploited which has been proposed in 2006 by C. Fermüller and R. Kosik. They, in turn, based on ideas already introduced by Robin Giles in the 1970th to obtain a characterization of $Ł_{\aleph_0}$ in terms of a Lorenzen style dialogue game combined with bets on the results of binary experiments that may show dispersion.

Keywords: Łukasiewicz's logic, quantum loigic, dialogue games, risk value

1 Introduction

In [4],[5] Robin Giles determines a logic for reasoning about physical theories with dispersive experiments, meaning that repeated trials of the same experiment may yield different results. Giles refers to Lorenzen's dialogue games for intuitionistic and classical logic which systematically reduce arguments involving logically complex assertions to arguments about atomic assertions.

In the issue Robin Giles formally defined a characterization of infinite-valued Lukasiewicz logic in terms of a game that combines dialogue rules for logical connectives with a scheme for betting on results of dispersive experiments for evaluating atomic propositions.

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In this connection it is interesting that Herman Dishkant introduced the modal extension of Łukasiewicz's infinite-valued logic which allows to consider physical objects obeying to the rules of quantum mechanics. This suggests to extend Giles' method to Dishkant's logic for obtaining a characterization of that in terms of a dialogue game too. The starting position and conditions in this case would be as follows.

The main idea of H. Dishkant's quantum modal logic (ŁQ) [1] is to include Mackey's axioms for probabilities of quantum-mechanical experiments [6] into the calculus of Łukasiewicz's infinite-valued logic $Ł_{\aleph_0}$ treating probabilities as truth-values. It is done not directly and Mackey's construction plays the role of a leading idea only and resulting calculus is, in essence, a modal extension of Łukasiewicz logic where the last is enriched with the modal symbol Q and four modal inference rules. The proposition QA expresses such a property which can be observed and the presence of which confirms A ('A is confirmed by observation').

The system ŁQ contains four axioms and five rules of inference:

A1. $A \to (B \to A)$

A2. $(A \to B) \to ((B \to C) \to (A \to C))$

A3. $((A \to B) \to B) \to ((B \to A) \to A)$

A4 $(\neg A \to \neg B) \to (B \to A)$

B5. $\dfrac{A, A \to B}{B}$

B6. $\dfrac{A}{QA}$

B7. $\dfrac{A}{\neg Q \neg A}$

B8. $\dfrac{A \to B}{QA \to QB}$

B9. $\dfrac{QA \to QB}{(QB \to QA) \leftrightarrow Q(QB \supset QA)}$

where $A \supset B =_{def} (\neg A \to B) \to B$.

Semantically Dishkant' system ŁQ would be interpreted in the following way. Usually a quantum object is described by a wave function — by a unit vector of a complex Hilbert space HR. Let Ψ be the set of all states of an object and besides these states we consider also *questions* which are described by closed subspaces of HR. Each such closed subspace $\hat{p}$ determines a probability $p(\psi)$ of a positive answer to the question for any $\psi \in \Psi$. It is known that this probability is equal to the squared modulus of the projection of ψ on the subspace $\hat{p}$, i.e. $p(\psi) = |\psi_{\hat{p}}|^2$.

Since $\hat{p}_1 \neq \hat{p}_2 \Rightarrow p_1 \neq p_2$ then we do not identify the question with the corresponding function of HR but with the corresponding function $p : \Psi \to [0, 1]$ for which there exists such $\hat{p}$ that $p(\psi) = |\psi_{\hat{p}}|^2$. Here $[0, 1]$ is the closed segment of real numbers.

Let $\mathcal{P}$ be the set of all questions and for any $p \in \mathcal{P}$ let $\hat{p}$ be the corresponding subspace of HR. We call any function $g : \Psi \to [0, 1]$ a generalized question and the set of all generalized questions will be denoted by $\mathcal{S}$. Obviously $\mathcal{P} \subset \mathcal{S}$. The set $\mathcal{S}$ is partially ordered by the relation $\leq$ which is defined by

$$g \leq h =_{def} \forall\psi(q(\psi) \leq h(\psi)) \textit{ for any } g, h \in \mathcal{S}.$$

Now let us fix a function $q : \mathcal{S} \to \mathcal{P}$ satisfying the conditions

q1. $g \leq h \Rightarrow q(g) \leq q(h)$

q2. $q(p) = p$

for any $g, h \in \mathcal{S}$; $p \in \mathcal{P}$. It is easy to see that there is at least such a function q (e.g. one may take $q(g)$ equal to that p for which $\hat{p}$ is the minimal subspace containing all $\psi \in \Psi$ for which $q(\psi) = 1$).

Any function $I^D : W^0 \to S$ is an interpretation of ŁQ if it satisfy the following conditions:

(I) $I^D(A \to B) = min(1, 1 - I^D(A) + I^D(B))$;

(II) $I^D(\neg A) = 1 - I^D(A)$;

(III) $I^D(QA) = q(I^D(A))$

for any $A, B \in W^0$, where W^0 — a set of formulas of ŁQ. Here $1 : \Psi \to \{1\}$, where Ψ is a set of all states of an object. It is obvious that I^D may be defined on V (the infinite list of propositional variables) arbitrarily and then extended uniquely on W^0, if q is fixed.

It seems that under such definition q plays for modal formules the same role as Mackey's function r which assigns to every triple (A, α, E) (where A is an observable, α is a state and B is a Borel subset of the real line) the number $r(A, \alpha, E), 0 \leq r(A, \alpha, E) \leq 1$. So, we can treat W^0 as the set of observables, $dom(S)$ as the set of states and $rng(S)$ as the set of all Borel subsets of the real line.

The following result holds for such an interpretation I^D[1, p. 152]:

THEOREM 1. *For any $A \in W^0$, if $\vdash A$ then $I^D(A) = 1$ for any intepretation I^D.*

The weak completeness (semantic correctness) of ŁQ was proved just relative to the usual quantum propositional logic QPL (by embedding QPL in ŁQ). In view of this the problem was formulated to construct semantic model like those of Kripke–Grzegorczyk but for ŁQ. In [7] such Kripke-type model for Ł$_{\aleph_0}$ was yielded where an accessibility relation is a ternary one and in [8, p. 67] such model was extended to ŁQ and the soundness and completeness of ŁQ in respect to those was proved.

According to [8] the ternary semantic of ŁQ would be described as follows. Ł-frame is a quadruple $\langle O, K, R, ^* \rangle$ where K is nonempty set of observation points (states), $O \in K$, R is a ternary accessibility relation on K and * — a unary operation on K. The following conditions for R and * are satisfied:

(p1) $ROaa$

(p2) $Raaa$

(p3) $R^2abcd \Rightarrow R^2acbd$

(p4) $R^2Oabc \Rightarrow Rabc$

(p5) $Rabc \Rightarrow Rac^*b^*$

(p6) $a^{**} = a$

(p7) $ROab \Rightarrow ROba$

(q1) $Rabc \Rightarrow Rbac$

(d1) $a < b =_{def} ROab$

(d2) $R^2abcd =_{def} \exists x(Rabx \& Rxcd \& x \in K)$.

A valuation v is defined as a mapping assigning the truth value from truth-value matrix for ŁQ to propositional variables in every point of K accounting the binary relation $<$ from (d1). An interpretation I is a natural extending of v on all formulas of $Ł_{\aleph_0}$ under condition that in any point of K the usual explication of connectives takes place. The formal definition is as follows:

a) v is a valuation in Ł-frame, i.e. v is a function $v : V \times K \to M_{[0,1]}$ (where $M_{[0,1]}$ is a logical matrix for $Ł_{\aleph_0}$ i.e. $M_{[0,1]} = \langle [0,1], \bar{\neg}, \mapsto, \{1\} \rangle$ where $\bar{\neg} x = 1 - x, x \mapsto y = \min(1, 1 - x + y)$. For any $p \in V$ and any $a, b \in K$ the following condition is satisfied:

(1) $a < b \& v(p, a) \neq 0 \Rightarrow (p, b) \neq 0$;

b) I is an interpretation associated with v, i.e. I is a function $I : W^0 \times K \to M_{[0,1]}$ satisfying for any $p \in V$, $A, B \in W^0, a \in K$ the following conditions:

(i) $I(p, a) = v(p, a)$;

(ii) $I(\neg A, a = 1 - x$ iff $I(A, a^*) = x$;

(iii) $I(A \to B, a) = \inf(1, 1 - x + y)$ iff for any $b, c \in K$, $Rabc$ and

$I(a, b) = x \Rightarrow I(B, c) = y$.

(iv) $I(QA, a) = 1$ iff for any $b \in K(ROab \Rightarrow \exists c \in K(RObc \Rightarrow I(A, c) \neq 0))$.

The following theorem was proved [8, p. 67]:

Theorem 2. *The system ŁQ is complete in respect to the ternary semantic that is for any $A \in W^0$, if $I(A) = 1$ for any intepretation I then $\vdash A$.*

We have the following finite model property.

The junction of both semantics of ŁQ can be achieved via putting for any $A \in W^0$, $dom(I^D(A)) \subseteq K$ and $rng(I^D(A)) \subseteq \{I(A,a) : a \in K\}$, that is, treating the set Ψ as K.

Proposition 1. *A formula F is valid in ŁQ if and only if F is valid in all those Ł-frames $\langle O, K, R,^* \rangle$ where K is finite.*

Proof. Let $\Pi = \langle O, K, R,^*, I\rangle$ and let $V_F = \{p_1, ..., p_n\}$ be the propositional variables occurring in F. Moreover, let $\mathcal{B}_F$ be the set of all bi-valued assignments $I_F : V_F \to \{0,1\}$. We write I_F^a if $\forall p \in V : I_F(p) = I(p,a)$ and define a new model $\Pi_f = \langle O', K_f, R',^{*\prime}, I'\rangle$ as follows:

- $K_f = \{I_F \in \mathcal{B}_F : \exists a \in K : I_F = I_F^a\}$
- $I'(p, I_F) = I(p,a)$,where $I_F = I_F^a$
- $R' \subseteq K_f \times K_f \times K_f$ where we take $R'I(a)I(b)I(c)$ as corresponding to $Rabc$.

We can uniquely extend this to all subsets of K_f. It is straightforward to check that $I(F,O) = I'(I_F^O, F)$. Thus we have shown that in evaluating F it suffices to consider Π_f with at most $2^{p(F)}$ where $p(F)$ is the number of different propositional variables occurring in F. □

The analysis shows that we can replace the rule (iv) with the rule $(iv)'$ without the loss of the generality :

$(iv)'$ $I(QA,a) = \inf\{I(A,c)$: for any $b \in K(ROab \Rightarrow \exists c \in K(RObc \Rightarrow I(A,c) \neq 0\}$.

Turning back to the game theoretic semantic of $Ł_{\aleph_0}$ it is worth to denote that recently some its extensions were obtained (cf. [2], [3]). It seems natural to adopt such an approach for producing this kind of semantics for ŁQ which is also, in fact, the specific extension of $Ł_{\aleph_0}$.

2 Dialogue Game for ŁQ

In 2006 C. Fermüller and R. Kosik [2] proposed some game theoretic interpretation for the SŁ system that extends both Lukasiewicz logic Ł$_{\aleph_0}$ and modal logic $S5$. It was builded on ideas already introduced by Robin Giles in the 1970*th* to obtain a characterization of Ł$_{\aleph_0}$ in terms of a Lorenzen style dialogue game combined with bets on the results of binary experiments that may show dispersion. In [2] the experiments were replaced by random evaluations with respect to a given probability distribution over permissible precisifications. We will try to implement main ideas of interpretation proposed (respectively modifying it) for obtaining game theoretical semantic for the ŁQ.

Assume that two players agree to pay 1€ to the opponent player for each assertion of an atomic statement, which is false in any $a \in K$ according to a randomly chosen set of observation points. More formally, given a set of all observation points K the *risk value* $\langle x\rangle_K$ associated with a propositional variable x is defined as $\langle x\rangle_K = I^D(x)$. In fact, $\langle x\rangle_K$ corresponds to the probabilities of having to pay 1€, when asserting x.

Let $x_1, x_2, ..., y_1, y_2...$ denote atomic statements, i.e. propositional variables. By $[x_1, ..., x_m||y_1, ..., y_n]$ we denote an *elementary state* in the game where the 1st — the first player — asserts each of the y_i in the multiset $\{y_1, ..., y_n\}$ of atomic statements and the 2nd — the second player — asserts each atomic statement $x_i \in \{x_1, ..., x_m\}$. The risk associated with a multiset $X = \{x_1, ..., x_m\}$ of atomic formulas is defined as $\langle x_1, ..., x_m\rangle_K = \sum_{i=1}^{m}\langle x_i\rangle_K$. The risk $\langle\rangle_K$ associated with the empty multiset is 0. $\langle V\rangle_K$ respectively denotes the average amount of payoffs that the 1st player expects to have to pay to the 2nd player according to the above arrangements if he/she asserted the atomic formulas in V. The risk associated with an elementary state $[x_1, ..., x_m||y_1, ..., y_n]$ is calculated from the point of view of the 1st player and therefore the condition $\langle x_1, ..., x_m\rangle_K \geq \langle y_1, ..., y_n\rangle_K$ (success condition) expresses that the 1st player does not expect any loss (but possibly some gain) when betting on the truth of atomic statements.

Now we accept the following dialogue rule for implication (cf. [2]):

($R_\rightarrow$) If the 1st player asserts $A \rightarrow B$ in point a then, whenever the 2nd player chooses to attack this statement by asserting A in point b, the 1st has to assert also B in point c (the points are choosing according to the condition (*iii*) above). (And *vice versa*, i.e., for the roles of 1st and the 2nd player switched.)

A player may also choose not to attack the opponent's assertions of $A \rightarrow B$. The rule reflects the idea that the *meaning of implication* entails the principle that an assertion of 'If A then B' obliges one to assert also B if the opponent in a dialogue grants (i.e. asserts) A.

The dialogue rule for the negation involves a relativization to specific observation points:

($R\neg$) If the 1st player asserts $\neg A$ in point a then the 2nd player chooses to attack this statement by asserting A in point a^*(the points are choosing according to the condition (*iii*) above). (And *vice versa*, i.e., for the roles of 1st and the 2nd player switched.)

The dialogue rule for the Q-modality also involves a relativization to specific observation points:

(RQ) If the 1st player asserts QA then the 1st also have to assert that A holds (its interpretation differs from 0) at any point that the 2nd may choose using the condition (*iv*) above (And *vice versa*, i.e., for the roles of the 1st and the 2nd switched.)

Henceforth we will use A^a as shorthand for 'A holds at the observation point a' and speak of A as a formula indexed by a, accordingly. Thus using rule (RQ) entails that we have to deal with indexed formulas also in rule ($R_\rightarrow$). However, we don't have to change the rule itself, which will turn out to be adequate independently of the kind of evaluation that we aim at in a particular context. We only need to stipulate that in applying ($R_\rightarrow$) the observation point index of $A \rightarrow B$ (if there is any) is used for defininig the respective indexes for the subformulas A and B. If, on the other hand, we apply rule (RQ) to an already indexed formula $(QA)^a$ then the index a is overwritten by whatever index b is chosen by the opponent player; i.e., we have to continue with the assertion A^b and, of course, we also have to account for indices of formulas in elementary states. We augment the definition of risk by $\langle x^a \rangle_K = 1 - I(x, a)$. In other words, the probability of having to pay 1€ for claiming that x holds at the observation point a is 0 if x is true at a and 1 if x is false at a.

We use $[A_1^{a_1}, ..., A_m^{a_m} || B_1^{b_1}, ..., B_n^{b_n}]$ to denote an arbitrary (not necessarily elementary) state of the game, where $\{A_1^{a_1}, ..., A_m^{a_m}\}$ is the multiset of formulas that are currently asserted by the 2nd player, and $\{B_1^{b_1}, ..., B_n^{b_n}\}$ is the multiset of formulas that are currently asserted by the 1st player. (We don't care about the order in which formulas are asserted.)

A move initiated by the1st player (1st-move) in state $[\Gamma || \Delta]$ consists in his/her picking of some non-atomic formula A^a from the multiset Γ and proceeding as follows:

- If $A^a = (A_1 \to A_2)^a$ then the 1st may either attack by asserting A_1^b in order to force the 2nd to assert A_2^c in accordance with $(R_\to)$, or admit A^a. In the first case the successor state is $[\Gamma', A_2^c || \Delta, A_1^b]$, in the second case it is $[\Gamma' || \Delta]$, where $\Gamma' = \Gamma - \{A^a\}$.
- If $A^a = (\neg A_1)^a$ then the 1st chooses the point a^* thus forcing the 2nd to assert $A_1^{a^*}$. The successor state is $[\Gamma, A_1^{a^*} || \Delta']$, where $\Delta' = \Delta - \{A^a\}$.
- If $A^a = QB^a$ then the 1st chooses an arbitrary $b \in K$ using the condition (iv) above thus forcing the 2nd to assert B^c. The successor state is $[\Gamma', B^c || \Delta]$, where $\Gamma' = \Gamma - \{A^a\}$.

A move initiated by the 2nd player (2-move) is symmetric, i.e. with the roles of the 1st and the 2nd players interchanged. A run of the game consists in a sequence of states, each resulting from a move in the immediately preceding state, and ending in an elementary state $[x_1^{a_1}, ..., x_m^{a_m} || y_1^{b_1}, ..., y_n^{b_n}]$. The 1st player succeeds in this run if this final state fulfills the success condition, i.e., if

$$\sum_{j=1}^{n} \langle y_j^{b_j} \rangle_K - \sum_{i=1}^{m} \langle x_i^{a_i} \rangle_K \leq 0.$$

The term at the left hand side of inequality is an *expected loss* of the 1st player at this state. In other words, the 1st succeeds if its expected loss is 0 or even negative, i.e., in fact a gain. The other connectives can be reduced to implication and negation.

3 Adequacy of the game

To show that the considered game indeed characterizes logic ŁQ, we have to analyse all possible runs of the game starting with some arbitrarily complex assertion by the 1st player. A strategy for the 1st player will be a tree-like structure, where a branch represents a possible run resulting from particular choices made by the 1st player, taking into account all possible choices of the 2nd player in (2- or 1-moves) that are compatible with the rules. We will only have to look at strategies for the 2nd player and thus call a strategy winning if the 1st player succeeds in all corresponding runs (according to condition (2)).

Taking into account that by Theorem (2) we can assume that the set K of observation points (states) is finite. The construction of strategies can be viewed as systematic proof search in an analytic tableau calculus with the following rules:

$$\frac{[\Gamma||\Delta,(A_1\to A_2)^a]}{[\Gamma,A_1^b||\Delta,A_2^c]\mid[\Gamma||\Delta]}(\to_{2nd})$$

$$\frac{[\Gamma,(A_1\to A_2)^a||\Delta]}{[\Gamma,A_2^c||\Delta,A_1^b]}(\to^1_{1st})\quad\frac{[\Gamma,(A_1\to A_2)^a||\Delta]}{[\Gamma||\Delta]}(\to^2_{1st})$$

$$\frac{[\Gamma||\Delta,(\neg A)^a]}{[\Gamma,A^{a^*}||\Delta]}(\neg_{2nd})\quad\frac{[\Gamma,(\neg A)^a||\Delta]}{[\Gamma||\Delta,A^{a^*}]}(\neg_{1st})$$

$$\frac{[\Gamma||\Delta,(QA)^a]}{[\Gamma||\Delta,A^{c_1}]|...|[\Gamma||\Delta,A^{c_n}]}(Q_{2nd})\quad\frac{[\Gamma,(QA)^a||\Delta]}{[\Gamma,A^c||\Delta]}(Q_{1st})$$

In all rules a can denote any index. In the rule (Q_{2nd}) as well as in the rule (Q_{1st}) we assume that indexes $c_1,...,c_n$ and c are defined by means of the condition (iv) above. Note that, in accordance with the definition of a strategy for the 2nd player, his/her choices in the moves induce branching, whereas for the 1st player choices a single successor state that is compatible with the dialogue rules is chosen.

THEOREM 3. *A formula F is valid in ŁQ if and only if for every set K of observation points (states) the 1st player have a winning strategy for the game starting in game state $[||F]$.*

PROOF. Every run of the game is finite. For every final elementary state $[x_1^{a_1}, ..., x_m^{a_m} || y_1^{b_1}, ..., y_n^{b_n}]$ the success condition says that we have to compute the risk $\sum_{j=1}^{n} \langle y_j^{b_j} \rangle_K - \sum_{i=1}^{m} \langle x_i^{a_i} \rangle_K$, where $\langle r^a \rangle_K = I(r, a)$ if $a \notin dom(I^D(r))$ and $\langle r^a \rangle_K = 1 - I^D(r)(a)$ otherwise, and check whether the resulting value (in the following denoted by $\langle x_1^{a_1}, ..., x_m^{a_m} || y_1^{b_1}, ..., y_n^{b_n} \rangle$) is ≤ 0 to determine whether the 1st player 'win' the game. To obtain the minimal final risk of the 1st player (i.e., his/her minimal expected loss) that the 1st can enforce in any given state S by playing according to an optimal strategy, we have to take into account the supremum over all risks associated with the successor states to S that you can enforce by a choice that you may have in a (2nd- or 1st-)move S. On the other hand, for any of the 1st player choices the 1st can enforce the infimum of risks of corresponding successor states. In other words, we prove that we can extend the definition of the 1st expected loss from elementary states to arbitrary states such that the following conditions are satisfied:

(3.1) $\langle \Gamma, (A \to B)^a || \Delta \rangle_K = \inf\{\langle \Gamma || \Delta \rangle_K, \langle \Gamma, B^c || A^b, \Delta \rangle_K\}$

(3.2) $\langle \Gamma, (\neg A)^a || \Delta \rangle_K = \sup\{\langle \Gamma || \Delta, A^{a^*} \rangle_K\}$

for assertions by the 2nd player and, for assertions by the 1st player:

(3.3) $\langle \Gamma || (A \to B)^a, \Delta \rangle_K = \sup\{\langle \Gamma, A^b || B^c, \Delta \rangle_K, \langle \Gamma || \Delta \rangle_K\}$

(3.4) $\langle \Gamma || \Delta, (\neg A)^a \rangle_K = \inf\{\langle \Gamma, A^{a^*} || \Delta \rangle_K \rangle\}$

Furthermore we have

(3.5) $\langle \Gamma || \Delta, (QA)^a \rangle_K = \sup\limits_{\substack{c \in K \\ ROab \Rightarrow RObc}} \{\langle \Gamma || \Delta, A^c \rangle_K\}$

(3.6) $\langle \Gamma, (QA)^a || \Delta \rangle_K = \inf\limits_{\substack{c \in K \\ ROab \Rightarrow RObc}} \{\langle \Gamma, A^c || \Delta \rangle_K\}$

We have to check that $\langle .||. \rangle_K$ is well-defined; i.e., that conditions above together with the definition of my expected loss (risk) for elementary states indeed can be simultaneously fulfilled and guarantee uniqueness. To this aim consider the following generalisation of the truth function for ŁQ to multisets G of indexed formulas:

$$I(\Gamma)_K =_{def} \sum_{\substack{A\in\Gamma \\ a\notin dom(I^D(A))}} I(A,a) \ + \sum_{\substack{A\in\Gamma \\ a\in dom(I^D(A))}} I^D(A)(a)$$

Note that

$$I(\{A\})_K = I(A)_K = \sum_{a\notin dom(I^D(A))} I(A,a) \ + \sum_{a\in dom(I^D(A))} I^D(A)(a) = 1 \text{ iff } \langle ||A\rangle_K \leq 0,$$

that is, A is valid in ŁQ iff my risk in the game starting with my assertion of A is non-positive. Moreover, for elementary states we have

$\langle x_1^{a_1},...,x_m^{a_m}||y_1^{b_1},...,y_n^{b_n}\rangle_K = n - m + I(x_1^{a_1},...,x_m^{a_m})_K - I(y_1^{b_1},..., y_n^{b_n})_K.$

We generalize the risk function to arbitrary observation states by

$$\langle\Gamma||\Delta\rangle^*_K =_{def} |\Delta| - |\Gamma| + I(\Gamma)_K - I(\Delta)_K$$

and check that it satisfies conditions (3.1)–(3.6). We only spell out two cases. In order to avoid case distinctions let $I(A^a)_K = I(A,a)$. For condition (3.1) we have
$\langle\Gamma,(A \to B)^a||\Delta\rangle^*_K = |\Delta| - |\Gamma| - 1 + I(\Gamma)_K + I(A \to B,a)_K - I(\Delta)_K = \langle\Gamma||\Delta\rangle^*_K - 1 + I(A \to B,a) = \langle\Gamma||\Delta\rangle^*_K - 1 + \inf\{1, 1 - I(A,b) + I(B,c)\} = \langle\Gamma||\Delta\rangle^*_K - 1 + \inf\{1, 1 + \langle B^c||A^b\rangle^*_K\} = \langle\Gamma||\Delta\rangle^*_K + \inf\{0, \langle B^c||A^b\rangle^*_K\} = \inf\{\langle\Gamma||\Delta\rangle^*_K, \langle\Gamma, B^c||A^b,\Delta\rangle^*_K\}$.

For condition (3.5) we have
$\langle\Gamma||\Delta,(QA)^a\rangle^*_K = |\Delta| - |\Gamma| - 1 + I(\Gamma)_K - I(\Delta)_K - I((QA)^a)_K = \langle\Gamma||\Delta\rangle^*_K + 1 - I(QA,a) = \langle\Gamma||\Delta\rangle^*_K + 1 - \inf\{I(A,c)$: for any $b \in K(ROab \Rightarrow \ \exists c \in K(RObc \Rightarrow I(A,c) \neq 0\} = \langle\Gamma||\Delta\rangle^*_K +$

$$\sup\{I(A,c)\text{:for any } b \in K(ROab \Rightarrow \exists c \in K(RObc \Rightarrow I(A,c) \neq 0\} = \sup_{\substack{c\in K\\ ROab\Rightarrow RObc}} \{\langle \Gamma || \Delta, A^c\rangle^*_K\} \qquad \square$$

Let us define a *regulation* as assignment of labels '*the 2nd player moves next*' and '*the 1st player moves next*' to game states that obviously constrain the possible runs of the game. A regulation is consistent if the label '*2nd*(*Ist*) *move next*' is only assigned to states where such a move is possible, i.e., where 1st player (2nd player) have asserted a non-atomic formula. As a corollary to our proof of Theorem (3), we obtain:

COROLLARY 1. *The total expected loss $\langle \Gamma || \Delta\rangle^*_K$ that the 1st player can enforce in a game over K starting in state $[\Gamma || \Delta]$ only depends on Γ, Δ and K. In particular, it is the same for every consistent regulation that may be imposed on the game.*

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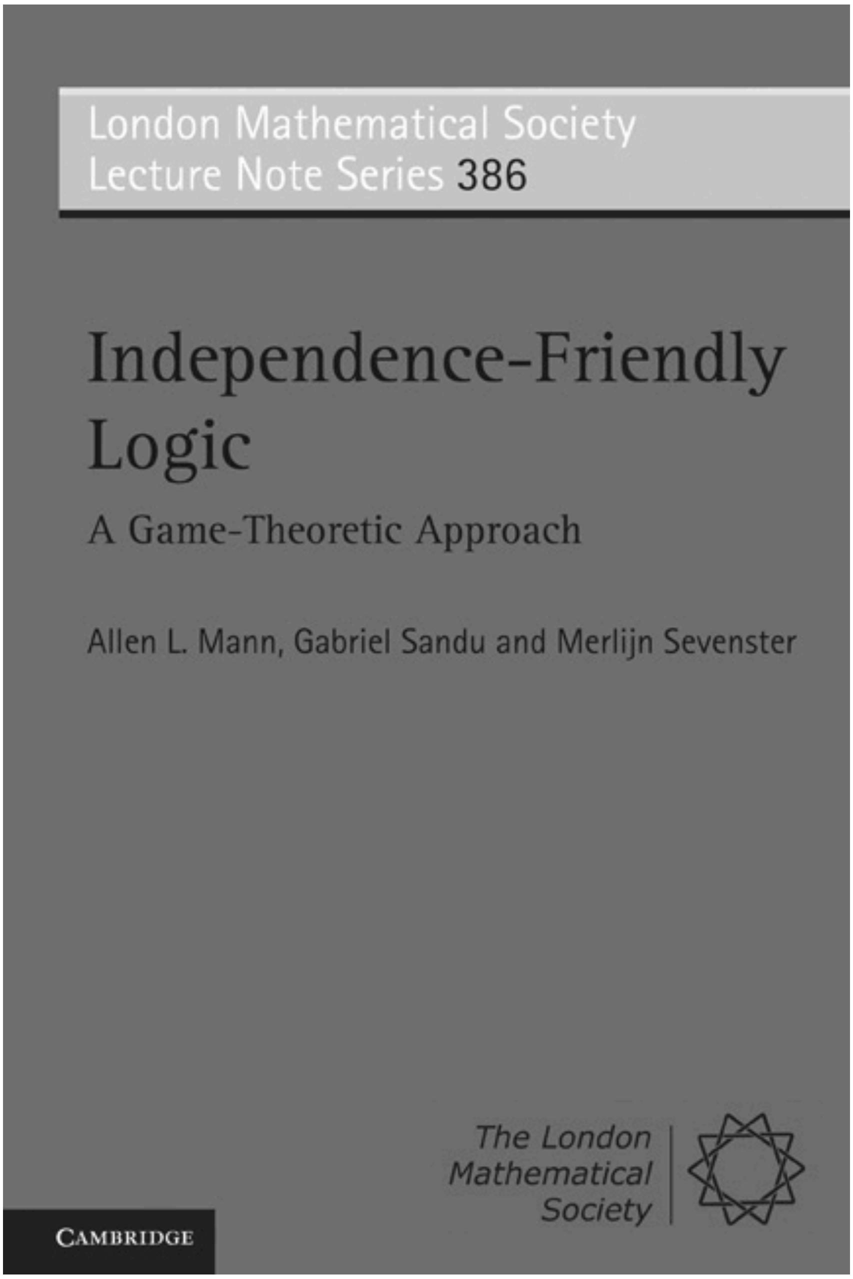
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